



LUND UNIVERSITY

Department of Computer Sciences

On a generalization of L_p
approximation and estimation

by
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This dissertation is based on the following two publications

I. On L_p -criteria, $p < 1$, for estimation and approximation.

Partially published in H.Ekblom and S.Henriksson: L_p -criteria for the estimation of location parameters.

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II. A generalization of L_p approximation to $p \leq 0$.

Preprint, to be submitted for publication.

This manuscript is bound in the following way

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Introduction and summary.

One of the more attractive features of applied approximation theory is the honesty with which it admits its subjective elements instead of clothing them in some verbal veil. Confronted with an approximation problem, the applied mathematician has to make choices. Given a function defined on some set, there are two essential points on which decisions have to be made.

One is the type of approximating function to be used. Should it be linear, rational or general nonlinear, which family of functions should be used: polynomials, exponentials, trigonometric functions, piecewise polynomials etc and what number of them should be used ?

The other one is the specification of the measure of the goodness of the approximation. What should constitute a good approximation? Should we allow sizable errors almost everywhere in order to bring down the maximum magnitude of the error or should we not care about some large errors if we have very small errors elsewhere ? Are there certain points or regions in which the error should be particularly small, necessitating the use of a weighted approximation ?

The answer to these and other questions are normally formed from several factors. The problem itself usually suggests that some approximating functions and some measures of goodness are more appropriate than others. Practical matters as availability of theoretical and computational tools influence the choice.

This dissertation has a connection to the second of the two strategic choices and develops some tools which can be used in that process.

Usually one there employs as a goodness-of-fit criterion the minimization of a mathematically well-defined norm or distance function. The class of L_p norms includes the most important and most frequently used norms and some of the great mathematicians are associated with the early history of their development: Laplace (L_∞), Legendre (L_2), Chebyshev (L_1, L_∞). However, in this era of upheaval against established norms, the time has come to the L_p norm.

A nice property of L_p -norms is the way their character is changed when the parameter p is changed. For $p=\infty$ we get the minimax or supremum criterion which minimizes the maximum error while errors below this do not count. As p goes down, errors elsewhere are increasingly taken into account, e.g. as when $p=2$ we are minimizing the squared errors. For $p=1$ we reach the situation that the magnitudes of the errors are unimportant; only the weighted sum or integral of the sign of the error is critical.

For some time no one went further down than $p=1$, one reason being that the L_p -norm loses many of its mathematically attractive features when $p < 1$, in fact no longer is a norm in the mathematical sense. It is not clear to the author who was actually first in treating the case $p < 1$, it may have been Jackson (1923), who showed the existence of best L_p approximations, $p > 0$, for bounded and measurable functions. An important paper is that of Hoel (1935) where three important steps were taken (in all cases for a continuous function, $p > 0$): best approximations fulfill a certain orthogonality condition, coefficients of best approximations are continuous functions of p , certain lower bounds on the multiplicities of the zeroes of the error function exist. The main work in this area, including the discrete case, was done by T.S. Motzkin and J.L. Walsh in a remarkable series of

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papers published during the 1950's and 60's, where they demonstrated the interpolatory properties of best L_p approximations and vice versa. J.R.Rice generalized their work in the direction of vari-solvent functions instead of Chebyshev systems as approximating functions and with a wider class of norms than L_p -norms, in a paper of 1960. Rice also gives an admirable summary of the results in the area in a book of 1969.

Of the present two papers, the first one started as a practical problem in an area that today would be called "data analysis". The problem was first presented to the author as a regression problem where a straight line was to be fitted to the data points in such a way that those measurement who where far away from the proposed line should not influence its exact placement. This lead the author to the question whether it was possible to go further in the L_p -family of norms or estimators, looking for a measure which not only did not care about the magnitude of large errors but which actually discriminated against data points producing such errors. The answer to the question was yes, as became clear from readings of the published literature in the area, dealing with the case $0 < p < 1$. Soon, however, in collaboration between H.Ekblom and the author, it became evident that it was possible to go even further with p in the discrete case, down to $p = -\infty$. The work as it was published, was formulated in terms of mathematical statistics, as estimators of location parameters (particularly the mode) of probability density functions, but it was clear that it could as well be used in the setting of approximation theory. The statistical results of the paper by Ekblom-Henriksson and hence paper I, are by themselves rather modest.

Paper II tries to develop the same idea as paper I but now in the

context of approximation theory. In the discrete case this is rather easy, on the other hand the results are not very interesting. For the interval case the situation is reversed. To overcome the difficulties we have restricted the discussion to approximation of analytic functions. Among the results here is that best approximations are full Lagrangian (Hermitian) interpolation functions for values of p from 1 down to a point in the interval $(-1,0)$ and that, as p approaches this value, the interpolation will take place in one point (with sufficient multiplicity of the root of the residual). For polynomial approximation this means that a link is provided by approximations that are good in some norm and those which are merely Taylor expansions.

It has puzzled me that the area $p \leq 0$ apparently has not been covered by the numerous earlier workers in this field. One possible explanation evolves from a paper by Dunham Jackson, one of the founders of modern approximation theory (Note on an ambiguous case of approximation, TAMS vol.24 (1923), pp 333-337). We cite the introduction in its entirety:

1. Introduction. Let $f(x)$ be a continuous function of period 2π . The writer has discussed to some extent the existence and properties of a trigonometric sum $T_{mn}(x)$, of order n at most, determined by the condition that the integral

$$(1) \quad \int_0^{2\pi} |f(x) - T_{mn}(x)|^m dx$$

shall have the smallest possible value, n being a given positive integer (or zero) and m a given number ≥ 1 , not necessarily integral.† The purpose of the present note is to inquire what becomes of the problem for values of m less than 1.

For $m < 0$, it is clear that the value of (1) can be brought arbitrarily near to zero by taking a sufficiently large constant for $T_{mn}(x)$, but can not be made equal to zero, so that the problem has no significance. For $m = 1$, the integral (1) is always equal to 2π , unless there is a trigonometric sum which is identically equal to $f(x)$ throughout a point set of measure different from zero, in which case the integral is not defined. Attention may therefore be restricted to values of m between 0 and 1.

We may disregard that the discussion in the paper is restricted to a trigonometric sum; it could just as well have been polynomials. It is easy to see that if Jackson instead had treated the minimization of the expression

$$\left[\int_0^{2\pi} |f(x) - T_{mn}(x)|^m dx \right]^{1/m}$$

his conclusions would have been different. The problem would have remained the same for $m > 0$, but for negative m the problem would have been the one of maximizing the integral. Hence the introduction of the L_p "norm" transforms the problem into a non-trivial one.

Finally, it should be noted that the two papers were written with a time interlude of three years. Thus the sketchy, optimistic and somewhat misleading part of I which deals with approximation should not be judged too harshly in view of II. The treatment of the subject matter is to some extent dialectic in paper I, the treatise moving from one point of view to another; it is hoped that the two papers together can be viewed in the same way: as documents from a research process.

Acknowledgements.

The author has had the privilege of having professor Carl-Erik Fröberg as his teacher and I want to thank him for his constant encouragement and support during my studies. His tolerance, openness and broadminded views on what should constitute an education in computer science has been a source of strength, of value to the author far outside the field of numerical analysis.

The present work is the result of collaboration, discussions, arguments etc. between Tåkan Ekblom and the author and any credit for the work should equally be shared between him and the author.

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May 1969

On L_p -criteria, $p < 1$, for
estimation and approximation.

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Contents.

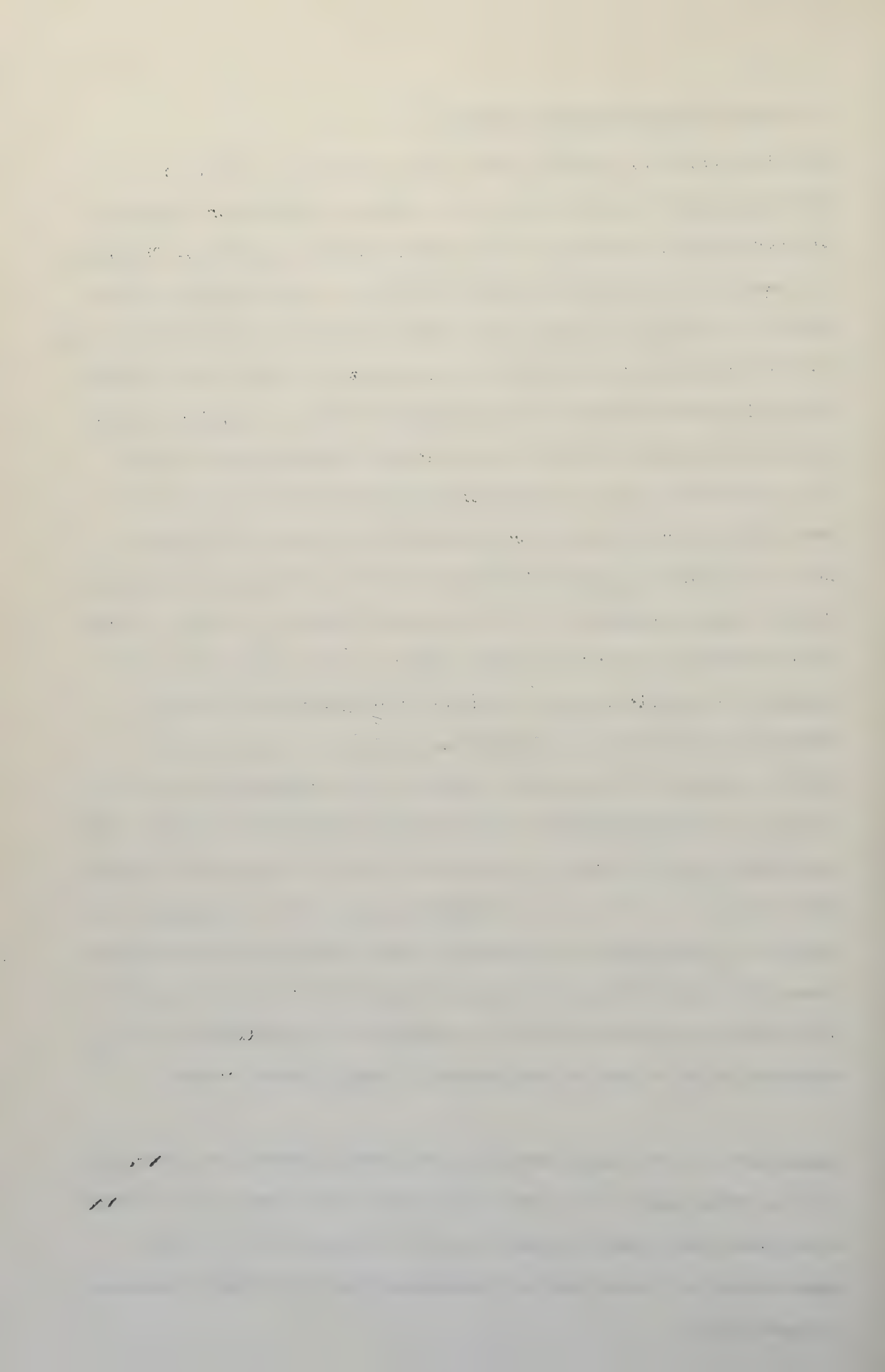
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	page
1. General introduction and summary	2
2. A problem of line fitting in presence of wild points	
2.1 Physical background	3
2.2 Formulation of the problem	5
2.3 Discussion of the one-parameter case	8
3. On estimation	11
4. Derivation of generalized L_p -estimation	15
5. L_p -estimation on symmetric distributions	
5.1 Experimental method	20
5.2 The distributions	20
5.3 The results	21
5.4 Discussion of results	26
6. The use of L_p -measures, $p < 1$, for approximation purposes	
6.1 Introduction	27
6.2 Approximation in the L_p -sense on a real finite point set	30
6.3 Approximation in the L_p -sense on a closed interval	31
6.4 Examples	35
References	38

1. General introduction and summary.

The study which is presented here was initiated by a specific line-fitting problem in connection with a computer program for evaluation of spark-chamber events in a high energy physics experiment. The physical background is discussed briefly in sect. 2. The presence of "wild points" ("outliers", points which evidently do not come from the studied normal distribution) gives rise to the question of what could replace the well-known criterion of least squares. Thus, the physical problem is replaced by the simpler and more abstract mathematical problem of estimating a location parameter. Also, the near relation between the outlier problem and the estimation of the mode is recognized. Sect. 3 contains a general discussion of the estimation problem. Least-squares being a member of the well-defined class of L_p -norms, it was found natural to search for other members of this class having the property of diminishing the influence of outliers. This led to a generalization of the L_p -minimization concept to $-\infty \leq p \leq +\infty$. This is derived and its general properties are discussed in sect. 4. In sect. 5 the experimental method of investigation is presented. The requirement that a method of estimation should be reasonably efficient when outliers in fact are not present was one of the reasons for the study of L_p -minimization efficiency of some symmetric frequency functions. The results from these computations are also given in sect. 5. We finish by discussing in sect. 6 whether the generalization of L_p -criteria could be used in the setting of approximation theory.

Large parts of this paper have been published by Ekblom and Henriksson in [1]. Other parts of [1] which only are touched upon or not covered at all here are treated by Ekblom in [2]. Generally [2] and this paper should be looked upon as interconnected, both being expansions of parts of [1].



2. A problem of line-fitting in presence of wild points.

2.1. Physical background.

In a high energy physics research project [17],[18], a problem of pattern recognition was to be solved. The problem is presented here in a highly simplified way, omitting the actual complicated circumstances.

Particles emitted from the CERN proton synchrotron were subject to studies in a spark-chamber set-up. The spark chamber (fig. 2.1.) contained a number of sections (≈ 5), each consisting of a number of plane-parallel plates (≈ 20), made of carbon or framed aluminium foil and separated by 10 mm gaps. When an event of the desired characteristics is recognized by scintillation detectors, a trigger circuit generates a high voltage pulse on each second plate. The open gaps between the plates are filled with a noble gas mixture

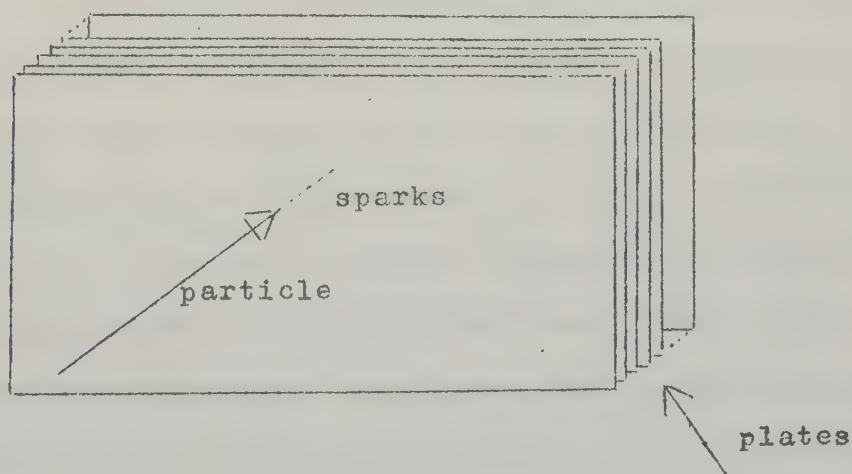


Fig. 2.1. Spark chamber section.

which becomes ionized when a charged particle passes through. Along the track of ionized gas a spark will then occur when the high voltage pulse is delivered. The sparks can easily be seen from positions which are perpendicular to the axis of the chamber and can be recorded, e.g. by a camera, if suitable optical arrangements are made.

Spark chambers have rather low resolution compared with bubble chambers but possess the advantage of high speed. Thus, it is easy to record say 10 events/sec which makes the spark chamber a suitable instrument in experiments where a large number of events are recorded. In the actual case the needed number of events was of the order of half a million, to get reliable statistics for the determination of a physical constant. It is imperative in such a case to use a computer and to have adequate, fully automated procedures.

Usually events in spark chambers are recorded on photographic films which later are inspected by humans and digitized in a semi-automatic procedure. Also various types of film-scanning machinery exist, in which the procedure is automated. The object of the project described here was to find out whether such photographic procedures could be replaced by a film-less technique. A specially designed TV-camera ("Vidicon") was installed by the spark chamber in parallel with photographic equipment. The signals from the vidicon tube were digitized and the digital recordings were either communicated to a computer operating on-line or stored on magnetic tape for future processing. The first task of the computer was to reconstruct the event as a number of straight line segments in a plane. The segments from various camera views were then put together in

a three-dimensional reconstruction of the event in terms of particle trajectories.

In a spark chamber event spurious sparks often occur. They do not represent true points along the trajectory of the particle. They may be caused by irregularities of the plates and intruding cosmic particles. Also errors in the optical system and electronic noise may cause readings which do not represent real sparks but which cannot easily be separated from such. This means that normally some "spurious sparks" (say max 10 %) will be present in the recording of an event. This indicates the need for methods which tend to diminish the influence of those recordings which in some sense are far from the bulk of observations.

2.2. Formulation of the problem.

In this subsection, the problem which has just been presented is formulated in a more abstract way. We will not return to the physical background, except for motivating certain restrictions and assumptions.

Generally, in this paragraph we assume the existence of a sample $\{x_i, y_i\}_{i=1}^m$, where the y_i 's are measured at certain points x_i which are fixed beforehand. We will now undertake a series of reformulations of the problem.

Formulation 1. The measurements are assumed to come from a random variable having a normal distribution $N(ax+b, \sigma)$. However, occasionally some disturbances occur, which will result in wild points. They

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15. 15. 15.

are rather rare and the overwhelming majority of the points do come from the normal distribution. Consequently, we can get the desired parameters a and b by the classical method of solving the 2×2 linear system of equations which are given by the normal equations.

Comment 1. This is obviously a very dangerous way of reasoning. Assumption of pure normality will lead to the use of the least squares method which places heavy emphasis on points which lie far from the main bulk of observations. The wild points will thus have an undue influence on the parameters. The existence of wild points implies that the distribution is not normal and we must reformulate the problem in order to have a model which includes their existence.

Formulation 2. We are observing a random variable with the frequency function f_1 (the one depending on the searched-for parameters) with the probability $(1-q)$ and from f_2 (the disturbing one, giving wild points) with the probability q . Thus we have the following frequency function

$$f = (1-q)f_1 + qf_2$$

As earlier we assume that f_1 represents the normal distribution while f_2 is unknown.

Comments 2. We have now constructed the frequency function as a sum of two components, one generating the points which carry information, the other one generating noise. The model is a very general one and will include a wide range of situations, from unperturbed normal distributions ($q=0$), to cases where the perturbing part more or less clouds the relevant information. To get a model which is

nearer to the physical reality which gave rise to the problem, it is reasonable to make some further assumptions, which are made in the next formulation of the problem.

Formulation 3. We are observing a variable with the following frequency function

$$f = (1-q)f_1 + qf_2, \quad \text{where } f_1(x,y) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-(y-ax-b)^2/2\sigma^2)$$

for all x and y

and $f_2(x,y)$ is a frequency function which is not known. We assume that, say $q < 0.1$, and that for each x , f_2 is such that if y' is the max point of f and y'' is the max point of f_1 , $y' = y''$. This means that the f_2 -component has no such peak that it reaches over the peak of the f_1 -component and that f_2 is constant in a neighbourhood of the max of f_1 .

Comments 3. We may now look at this formulation of the problem in the light of the physical problem presented earlier. There, measurement was restricted to a finite region, a rectangle in the xy -plane while f_1 here is defined for all x and y . However in the actual case, σ can be regarded as small in comparison with the width and breadth of the rectangle, and all points generated by the f_1 -component can be relied upon to be well clear of the boundaries of the region. The assumption regarding the peak(s) of f_2 is realistic in the sense that if some special point of irregularity in the experimental arrangements persistently generated "spark"-recordings, this could be corrected. Thus we conclude that the approximations and assumptions which have been made are moderate and that the model is

a fairly realistic one. In fig. 2.2 is a picture of a sample from a distribution similar to the one of the model.

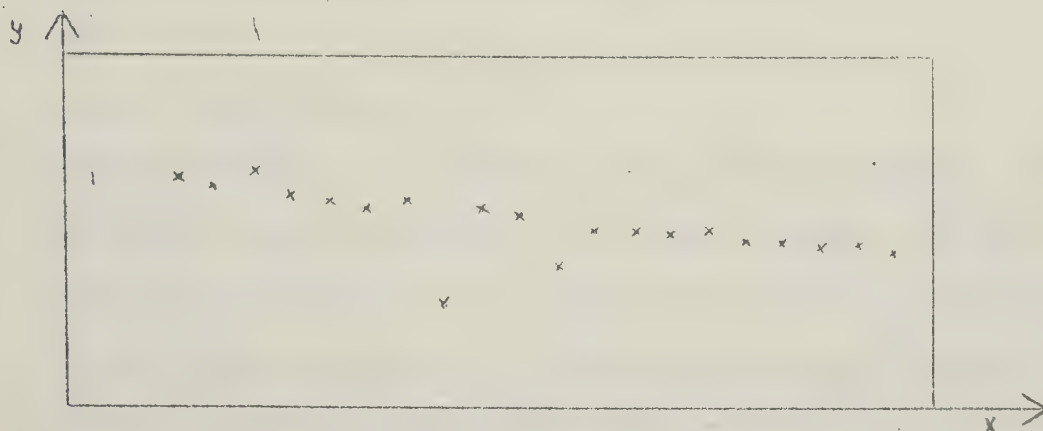


Fig. 2.2. A sample from the distribution as of formulation 3.

We will now leave the just formulated case of non-normal regression. Instead we will concentrate on the one-dimensional, one-parameter case. The main reason for this is that estimation of several parameters will involve very extensive computations when experimental methods are used as will be the case here. Further, results from an earlier investigation, [3], seem to show that no striking differences occur between results from the one-dimensional to the regression case. It may be enough to say that the one-parameter case is far from easy in itself and that it has to be handled before methods for regression can be developed.

2.3. Discussion of the one-parameter case.

We will now make another reformulation of the problem as it will be treated here.

Main formulation.

We have the ordered sample $\{y_i\}_{i=1}^m$, where $y_i \leq y_{i+1}$. The y_i 's are independent observations of the random variable y with

distribution function $F(y)$ which depends on the location parameter a . The frequency function for $F(y)$ is $f(y)$ and it has the following form

$$f(y) = (1-q)f_1(y) + qf_2(y)$$

We assume that f_1 is a symmetric and unimodal frequency function, the form of which is known to us, while its mean a is not. The frequency function f_2 is not known while certain assumptions can be made. The parameter q is approximately known. Further, $y' = y''$, where y' is the max point of f while y'' is the max point of f_1 .

We may now regard the situation in at least two different ways.

A. Our task is to estimate the mean of f_1 . To be able to do that, we must somehow identify those observations which have not been generated by the f_1 -component. Such points should be eliminated from the calculations while the rest could be used for estimating the mean of f_1 .

B. We cannot reliably separate the two components of the observations and consequently we should consider the sample in its entirety. Our task is to estimate the mode of f , which is almost identical to the mean of f_1 , if modest assumptions are made of f_2 .

Approach A is the method generally used and it leads to methods like trimming and winsorizing. These methods are well known and extensively treated in the literature, e.g. see [20] and references contained there.

Approach B to some extent means that our position in solving the problem is weakened. One may say generally that by treating the situation as a general problem of estimation of the mode we cannot utilize all of the information which is at our disposal, if we have further knowledge of the disturbing component. On the other hand, in the absence of such information we do not have to make such restricting assumptions as in A.

Here we will treat approach B and in fact a very special part of it. The general mode problem is treated extensively in [2] as are the possibilities of using L_p -methods for mode estimation. We will turn to a special aspect of this problem, the question of the behaviour of L_p -estimates when in fact no disturbing component is included, i.e. $q=0$ in the expression for f . Besides, an investigation of the generalized L_p -estimate which is to be defined in sect. 4, may have some value in itself.

3. On estimation. =====

The use of the word estimation will here be restricted^{*)} to the following well-known procedure of mathematical statistics.

Let $\{y_i\}_{i=1}^n$ be a number of observations of a random variable η with the distribution function $F_\eta(y, a)$, a being an unknown parameter. We want to estimate the value of a by a value a^* which is to be derived from the observations in such a way that it in some sense is near to the true value. It is then customary to look at a^* from the following aspects.

A. Consistency.

a^* is consistent if to each $\epsilon, \delta > 0$ there exists an m' such that $P(|a - a^*| < \epsilon) > 1 - \delta$ for all $n > m'$, where n denotes the number of observations. This is often expressed by saying that the estimate converges in probability to the true value.

B. Bias.

a^* has no bias if $E(a^*) = a$. The mean of an unbiased estimate is equal to the true value of the parameter.

C. Efficiency.

a^* is efficient if $E(\|a - a^*\|)$ is as small as possible. Here $\|\cdot\|$ denotes some measure of deviation, usually a distance function.

*) The connection between estimation and approximation is discussed by Rice in [5]. In his paper rigorous and rather general definitions are given of the two problems and it is argued that estimation rather than approximation is the relevant problem in numerical analysis and "approximate mathematics".

Often the concept of efficiency is based upon $E((a-a^*)^2)$ which is proved to have a lower bound in the Cramér-Rao inequality.

D. Economy.

The method for the computation of a^* should be well suited to the observational and computational means at disposal.

E. Robustness.

The conditions A, B and C should be reasonably well satisfied even if there are slight deviations from the model from which the estimate is derived.

Out of these, the aspects D and E are clearly of another category than the others, being less precise. Often demands of these types are not even explicitly stated. The classical method of getting an estimator has been to state various demands of type A, B and C (often with compromises between demands of type B and C). With suitably chosen distributions this procedure will give estimates which are computationally convenient too. Further interest in the aspects of computational and observational convenience has led to developments like range statistics and linear estimates. Lately, robustness has been stressed leading to estimates where the other demands have to give concessions.

Thus, what is considered to be a "good" estimate is far from invariant with time since various demands are stressed during different periods. It is obvious that the advent of the electronic computer must have a large influence in this area, making methods

which earlier were considered cumbersome and impractical more attractive.

The approach that is taken here is to some extent in contrast to the customary one of mathematical statistics. There, one usually starts by building a model, assuming the nature of the frequency function and from that deriving methods of estimation which reasonably satisfy the demands A,B and C and take into account D and E. We have used this way in order to motivate our interest in the estimation of the mode. However, the main emphasis can be changed to that of investigating the properties of a technique, generalized L_p -minimization. We will look at its behaviour for various distributions. Its use is suggested in situations when very little is known of the underlying distribution of a set of observations, thus making maximum-likelihood estimation and other estimators based on a prior knowledge of the frequency function unreasonable.

That such general techniques are of interest can easily be argued, e.g. from the praxis of a university computing center. In the experience of this author it happens very often that researchers with observational material come to get it evaluated without any strong convictions as to the underlying distribution. In the general vagueness of the situation, one often resorts to the well-known methods of least-squares. This was quite natural when other criteria led to staggering computational difficulties, but such is not the case with a modern computer.

We will find angry criticism of the prevailing L_2 -choice in [3],

and the following discussion of "the principle of least squares" is well worth quoting: " This 'principle' is usually presented (though often only implicitly) as the correct norm for smoothing and estimation. It is probable that this particular choice has been elevated to the position of a principle to disguise the fact that there is no single correct criterion and even if the distribution of the e_i (errors/SH) is known it is generally impossible to determine the 'best' L_p -norm (much less the best norm from some larger class of norms). Nevertheless the L_2 -norm is consistently presented (and usually unconditionally accepted by those who do not give serious thought to the question) and other norms are consistently ignored in the analysis of linear smoothing, estimation and regression."

R.W.Hamming in [15] delivers another salvo: " The criteria (least squares/SH) is popular with mathematicians because of the beauty of the mathematical theory which goes with it and with physicists because they believe that least squares is a principle of nature, but a person in the computing field is entitled to view the criterion with suspicion in any particular application."

The gallant spirit of these remarks has very much inspired the present investigation.

4. Derivation of generalized L_p -estimation.^{*)}

The L_p -estimation problem can be stated in the following way for the estimation of a location parameter.

Let $y_1, y_2, \dots, y_m$ be an ordered sample ($y_i \leq y_{i+1}$) and

$y_0 = -\infty$, $y_{m+1} = +\infty$ for convenience.

We call a_p the L_p -estimate of $\{y_i\}_{i=1}^m$ if $a = a_p$ minimizes the L_p -norm

$$(4.1) \quad L'_p(y-a) = \left[\sum_{i=1}^m |y_i - a|^p \right]^{1/p}$$

It is well known that for $1 \leq p \leq \infty$, $L'_p(y)$ defines a norm and that a_1, a_2 and a_∞ correspond to the median, the mean and the mid-range points of the data. We get a unique solution except for the case $p=1$ and m even, where uniqueness is attained in the usual way by taking the average of the two extremal candidates.

We will now insert a factor $1/(m-1)$ which generally is not used in the definition of the L_p -quantity. However, it will not change the minimum point for $p \geq 1$ and it will facilitate a definition of L_p -minimization for all p . Thus

$$(4.2) \quad L_p(y-a) = \left[\sum_{i=1}^m |y_i - a|^p / (m-1) \right]^{1/p}$$

Leaving the interval $p \geq 1$, $L_p(y)$ no longer defines a norm. For $0 < p < 1$, the triangle inequality is not valid. Following others, we may then call $L_p(y)$ a quasi-norm. For negative p , the requirement $L_p(y) = 0$ iff $y_i = a$ for all i is not satisfied either. In this case the equivalence is true if all is replaced by some, implying

^{*)} This section has been written in near collaboration between H. Ekblom and the author and it is substantially identical to a section in [1].

that one of the points y_i must be taken as the estimate. This latter property is in fact valid for $0 < p < 1$ -estimates too. To show this we look at $(L_p(y))$ as defined in (4.2). It is a continuous function of a and so is $(L_p(y))^p$ which is easier to handle and has the same minimum points. The second derivative of this,

$$(4.3) \quad \frac{\partial^2 ((L_p(y))^p)}{\partial a^2} = \frac{p(p-1)}{(m-1)} \sum_{i=1}^m |y_i - a|^{p-2}, \quad a \notin \{y_i\}_{i=1}^m, \quad 0 < p < 1,$$

is negative in every open interval (y_i, y_{i+1}) , $i=0, 1, \dots, m$.

This implies that the minimum must be achieved at one of the endpoints of the intervals.

Consequently, a can be replaced by y_k , minimizing over k . As all sums then have a term $|y_k - y_k|^p$, this can be left out. For negative p this is a necessary operation and gives a workable definition in a natural way.

Now the results for "elementary mean values" in [13] and [14] can be applied in order to obtain the limit cases $p=0$ and $p=-\infty$. Taylor expansion (as used in [13]) or an application of l'Hospital's rule to the logarithm of the quantity (as indicated in [14]) give the limit for $p=0$

$$(4.4) \quad \min_k \frac{1}{n} \sum_{i \neq k} |y_i - y_k|$$

For p with a large negative value, suppose that each y_k has a unique point y_j as its closest neighbour (j obviously equals $k-1$ or $k+1$), that is for all k , there are j 's such that

$$(4.5) \quad |y_j - y_k| < |y_i - y_k| \quad \text{for all } i \neq j, k. \quad \text{Now,}$$

$$(4.6) \quad \min_k \left[\sum_{i \neq k} |y_i - y_k|^p / (m-1) \right]^{1/p} \quad \text{can be written as}$$

$$(4.7) \quad \min_k \frac{|y_j - y_k|}{(m-1)^{1/p}} \left[1 + \sum_{i \neq j, k} |(y_j - y_k)/(y_i - y_k)|^{-p} \right]^{1/p}$$

According to (4.5) all terms in the sum are less than 1 and when $p \rightarrow -\infty$ we get

$$(4.8) \quad \min_{k, j} |y_j - y_k|$$

which may be written in the form

$$(4.9) \quad \min_k (y_{k+1} - y_k) \quad .$$

Supposing that we have reached a minimum for $k=K$, we have two candidates for the estimate, namely y_K and y_{K+1} . To select one of them we inspect the quantity (4.6) which is to be minimized. After dropping the term $(y_{K+1} - y_K)^p$ common to both sums, the limit value as $p \rightarrow -\infty$ can be derived in the same way as above. They will be $\min(y_K - y_{K-1}, y_{K+2} - y_K)$ for $k=K$ and for $k=K+1$. Thus, the point y_K is chosen if $(y_K - y_{K-1}) < (y_{K+2} - y_{K+1})$ and y_{K+1} otherwise.

Summarizing the various cases we get

Definition: Generalized L_p -smoothing and estimation. The number a_p is the L_p -estimate of the sample $\{y_i\}_{i=1}^m$ if a_p minimizes the L_p -measure

$$a) \quad p \geq 1 \quad \left[\sum_{i=1}^m |y_i - a_p|^p / (m-1) \right]^{1/p}$$

$$b) \quad 0 < p < 1 \quad \left[\sum_{i \neq k} |y_i - y_k|^p / (m-1) \right]^{1/p} \quad a_p = y_k$$

$$c) \quad p = 0 \quad \min_{i \neq k} |y_i - y_k| \quad a_p = y_k$$

$$d) \quad -\infty < p < 0 \quad \text{the same quantity as in b) } a_p = y_k$$

$$e) \quad p = -\infty \quad (y_{k+1} - y_k) \quad a_p = \begin{cases} y_{k+1} & \text{if } (y_{k+2} - y_{k+1}) < (y_k - y_{k-1}) \\ y_k & \text{otherwise} \end{cases}$$

Of these, a) is well known, b) has been discussed e.g. in [3], e) is similar to a type of estimates arrived at "intuitively" by Dalenius and treated in [2].

It is obvious that more precise definitions can be made in b)-e) in order to avoid deadlocks between possible candidates y_k . Special rules can be constructed using the same ideas which have been applied earlier. Such further considerations have been left out in order to get a simple definition for the numerical experiments. E.g. if two or more sample points coincide, they should obviously be chosen as the estimate for all $p \leq 0$. In view of the low probability of getting non-unique solutions in experimental work with 10-digit accuracy (as used) the definition above has been used and if several equally good estimates in fact have existed, the one first encountered has been taken although some other procedure might have been more appropriate.

The expressions which are to be minimized in b) and d) can easily be changed to more convenient forms. Thus, in b) it is enough to seek $\min_k \sum_{i \neq k} |y_i - y_k|^p$, while in d) it is equivalent to seek $\max_k \sum_{i \neq k} 1/|y_i - y_k|^{-p}$.

Example. In fig 4.1 the shape of the curve $L_p(y, a)$ is shown for $p=0.5$, $y=\{1, 2, 4, 5, 8\}$. We get a minimum for $a=4$.

($L_p(y, a)$ here denotes $\sum_i |y_i - a|^p$)

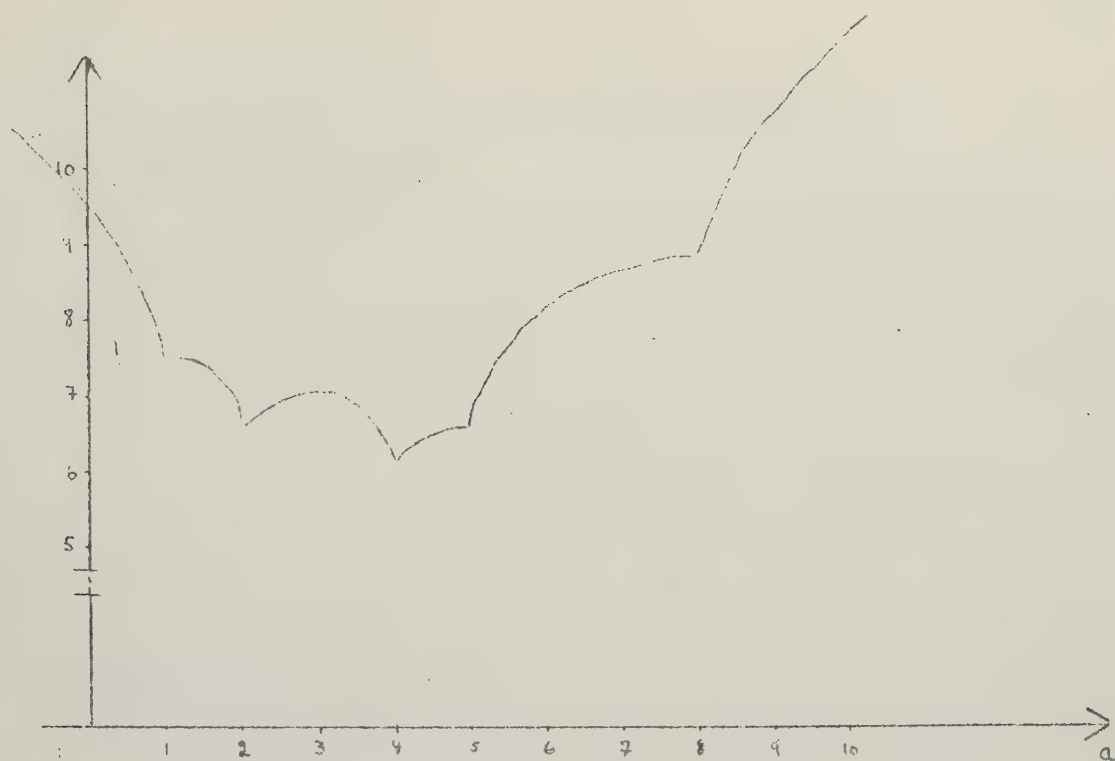


Fig.4.1. $L_p(y, a)$ as a function of a for $p=0.5$, $y=\{1, 2, 4, 5, 8\}$.

5. L_p-estimations on symmetric distributions.

5.1. Experimental method.

The general procedure of the computation was to generate m pseudo-random numbers from the distribution of current interest and to compute L_p -estimates of the type defined in the preceding section. This procedure was repeated n times and the dispersion measures of the results were calculated and printed.

The size of the experiments was limited by economical resources. To be able to get a representative picture of the dispersion as a function of p without too much computation, the following values of p were used: $\pm\infty, \pm 2, \pm 1, \pm 0.5$ and 0 . Since the computing time was roughly proportional to nm^2 , it suggested the following combinations:

m	5	10	20	40	80
n	800	800	100	100	20

Samples smaller than 5 points are not of very great interest since sample sizes of 3 and 4 give the same estimate for all $p \leq 1$ and $p < 1$, respectively.

The program was written in ALGOL and run on the 12-year old SMIL, the oldest computer in use in Sweden, with a total computation time of about 80 hours.

5.2. The distributions.

The symmetric distributions used in the experiments are unimodal, which means that the mean (when it exists), the median and the mode coincide. The choice of the distributions for consideration is somewhat arbitrary. The normal distribution was selected because of its fundamental role in statistics. The graphs of Rice and White in [3] inspired the choice of the Laplace distribution

$$f(x) = \exp(-|x|)/2$$

and the Cauchy distribution

$$f(x) = (1/(1+x^2))/\pi$$

since the standard deviation of the estimates a_p from these two distributions were decreasing as p went from $+\infty$ to 1 which could possibly go on further for smaller p . Finally, the following distribu-

tion was constructed

$$f(x) = \exp(-|x|^{1/4})/4e$$

to get an idea of the behaviour of a distribution for which some L_p -estimate with $p < 1$ was better than with $p \geq 1$.

To get an idea of the precision of the results obtained the following considerations have been made. For a given p , each sample will give an estimate a_p^i , $i=1,2,\dots,n$. From the symmetry of the distributions it follows that all these estimates are unbiased (for all p). Thus for the distributions considered, the expected value is zero,

$$E(a_p^i) = 0$$

Now, the parameter of interest is the standard deviation of the distribution of a_p^i and it is denoted by σ_p .

$$D(a_p^i) = \sigma_p^2$$

This parameter is estimated by the standard deviation of the observed values a_p^i , denoted by $\hat{\sigma}_p$:

$$\hat{\sigma}_p^2 = \frac{1}{n-1} \sum_{i=1}^n (a_p^i - \bar{a}_p)^2$$

Assuming the points a_p^i to come from a normal distribution, confidence intervals can be constructed by using the following properties of $\hat{\sigma}_p$ (see e.g. Cramér [15] p.353):

$$E(\hat{\sigma}_p) = \sigma_p + O(1/n), \quad D^2(\hat{\sigma}_p) = \sigma_p^2/2n + O(1/n^2)$$

This implies that $\hat{\sigma}_p/\sigma_p$ is asymptotically normally distributed with mean 1 and standard deviation $1/\sqrt{2n}$. Approximate 95%-confidence intervals for σ_p are thus given by $(\hat{\sigma}_p/(1 \pm 1.96/\sqrt{2n}))$. For the values used this gives

m	n	95%-confidence intervals of σ_p
5,10	800	$(0.95\hat{\sigma}_p, 1.05\hat{\sigma}_p)$
20,40	100	$(0.88\hat{\sigma}_p, 1.16\hat{\sigma}_p)$
80	20	$(0.76\hat{\sigma}_p, 1.45\hat{\sigma}_p)$

5.3. The results.

The results of the experiments is presented in fig. 5.1 - 5.4. The standard deviation is plotted as a function of p for the four distributions described in the preceding section.

Fig. 5.1. The standard deviation as a function of λ for the normal distribution.

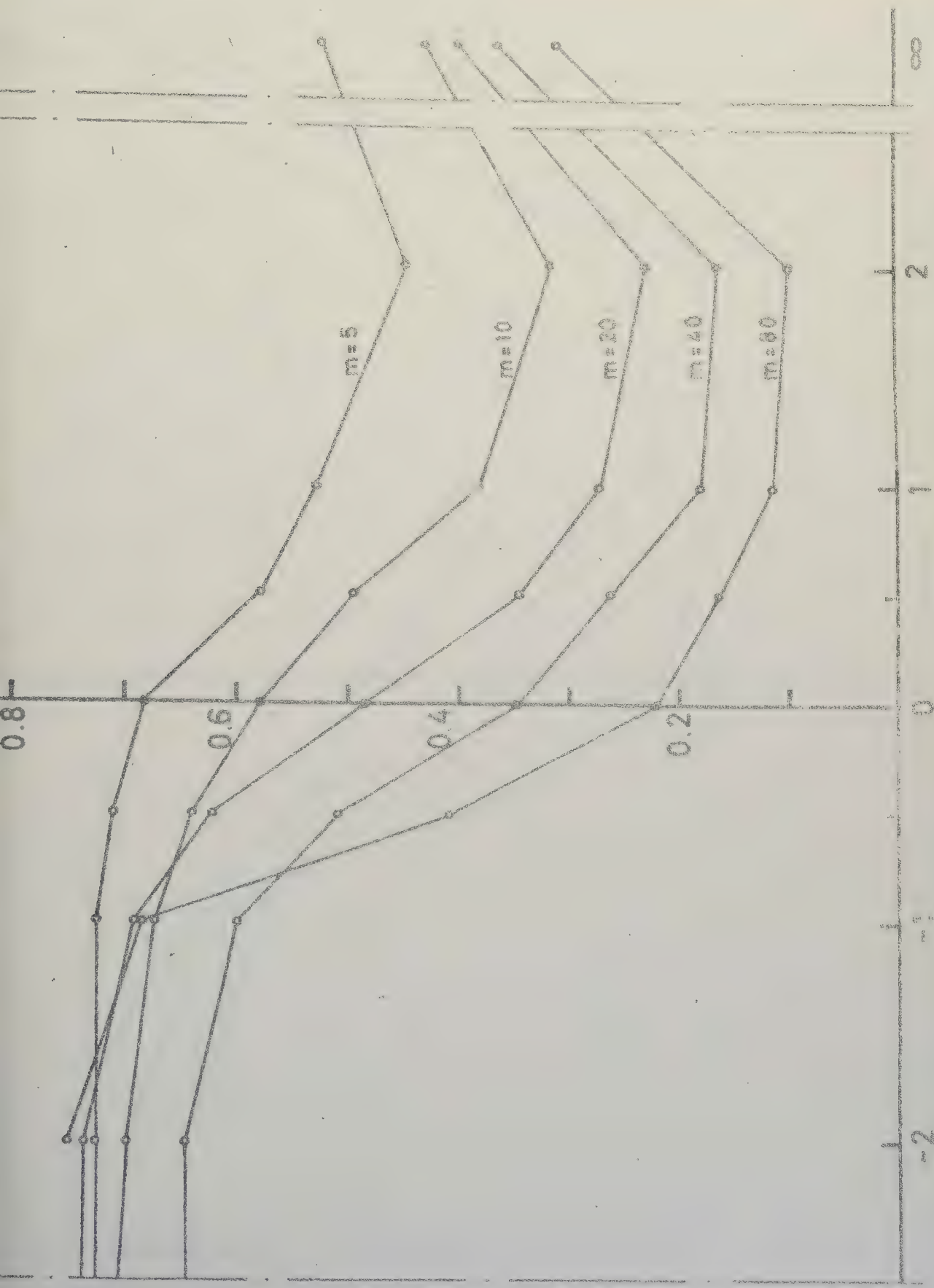


Fig. 5.2. The standard deviation as a function of p for the Laplace distribution.

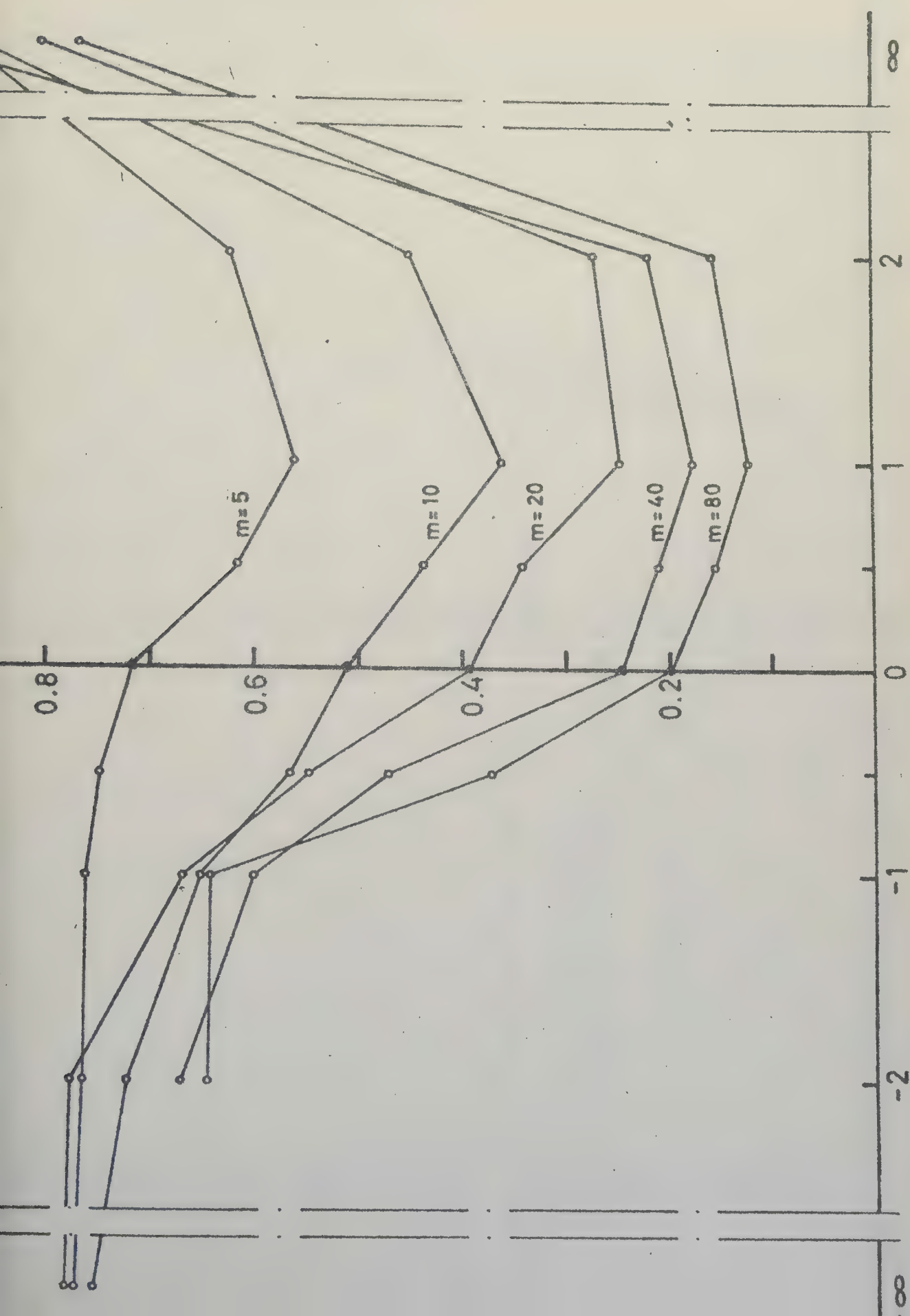


Fig. 5.5. The standard deviation as a function of p for the Cauchy distribution.

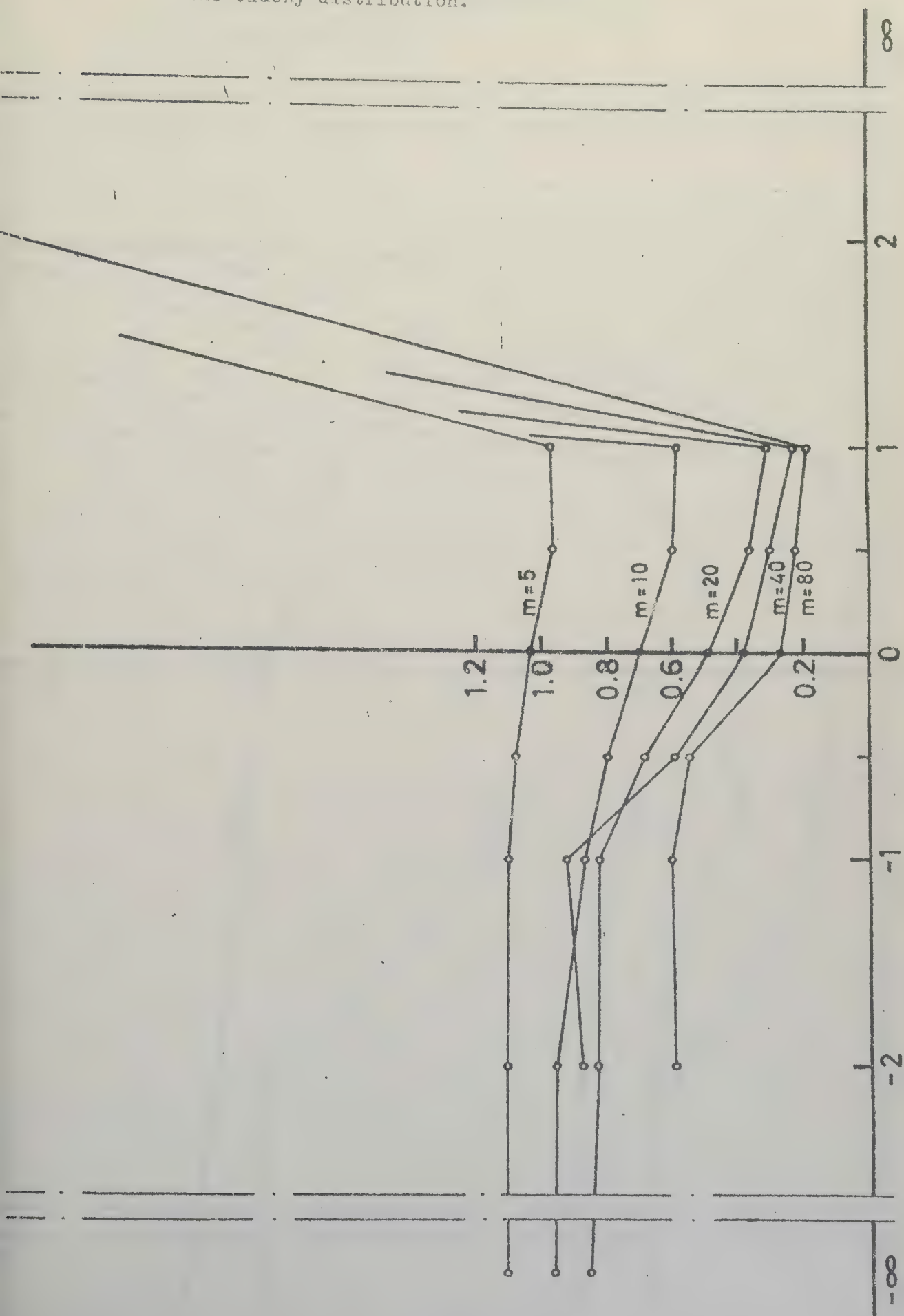
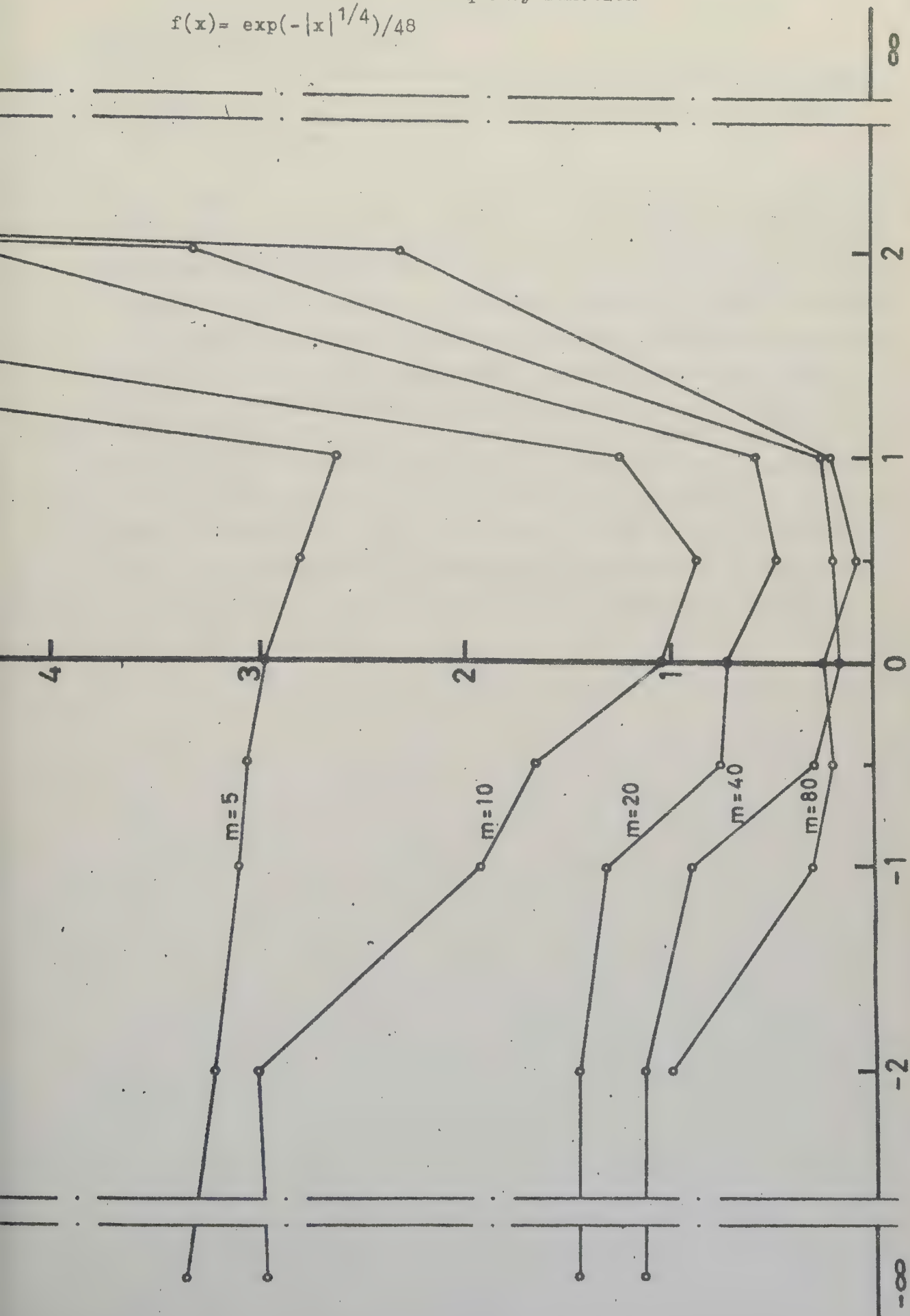


Fig. 5.4. The standard deviation as a function of p for the distribution with frequency function $f(x) = \exp(-|x|^{1/4})/48$



5.4. Discussion of results.

The purpose of the investigation was to get a picture of the general behaviour of L_p -estimates for $p < 1$, with sideglances to the possibilities of practical usage.

The results show that these estimates usually can not compete with the traditional ones as estimates of the mean-median-mode parameter in symmetric distributions. For the specially constructed distribution, with results in fig. 5.4, we get a minimum for $p < 1$ as expected but for the others the dispersions are rising, making $p < 1$ -estimates less attractive. On the other hand, they do not rise very rapidly. E.g. for the normal distribution, the L_0 -estimate has roughly the same efficiency as L_∞ , a criterion which is actually used in some observational situations. Here we may also note that if we want to use a L_p -estimate with $0 < p < 1$ for reasons of robustness, we shall not get any catastrophic loss of efficiency if the normal distribution is in fact unperturbed. For the Cauchy distribution, L_1 may be the best L_p -estimate, but the curves are here remarkably constant as p goes from 1 to $-\infty$.

6. The use of L_p -measures, $p < 1$, for approximation purposes.

6.1. Introduction.

In this section we will discuss whether the notion of generalized L_p -minimization could be used in the context of approximation.

We will here state the approximation problem as follows. Let f be a function defined on the set X , which is either a finite point set or an interval on the real line. Let H be an n -dimensional linear subspace of functions defined on X . We seek to approximate f by an element of H .

If d is a metric and if g^* is an element of H such that $d(f, g^*) \leq d(f, g)$ for all $g \in H$, then g^* is termed a "best d -approximation of f from H " or simply a best approximation.

A commonly used class of metrics is the family of L_p -norms, defined for $1 \leq p \leq +\infty$.

X	Def. of $d(f, g)$
finite point set	$\left[\sum_{x \in X} f(x) - g(x) ^p \right]^{1/p}$
interval	$\left[\int_{x \in X} f(x) - g(x) ^p dx \right]^{1/p}$

If 0 is the only function in H which possesses n or more roots in X , the H is termed a Haar space. An equivalent property is that for every selection of n distinct points $x_1, x_2, \dots, x_n \in X$, the determinant

$$\begin{vmatrix} h_1(x_1) & \dots & h_n(x_1) \\ \vdots & & \vdots \\ h_1(x_n) & \dots & h_n(x_n) \end{vmatrix}$$

is different from zero. When H is a Haar space, the general behaviour of best approximations in the L_p -norms is well known with respect to uniqueness, characterization and algorithms for solution, see e.g. [17].

However, as in the context of estimation, the concept of L_p -minimization can be extended to $p < 1$. In fact, the case $0 < p < 1$ has

been treated by Motzkin and Walsh in a series of papers, [6]-[11], and by Rice in [12]. In the construction of a theory, they have made use of the interpolating properties of best approximations.

A function g defined on a real finite point set X is said to be a strongly interpolating function with n strong sign changes if there exist points $x_0, x_1, \dots, x_n \in X$ such that $x_0 < x_1 < \dots < x_n$ and $g(x_i) \cdot g(x_{i-1}) < 0$ for $i=1, 2, \dots, n$.

If the last inequality is replaced by $g(x_i) \cdot g(x_{i-1}) \leq 0$, then g has n weak sign changes and it is called a weakly interpolating function. If $g(x_i) = f(x_i)$ for $i=1, 2, \dots, n$, call g an exactly interpolating function of f .

If X is an interval, then we say that the function g strongly interpolates to f if $f-g$ has n strong sign changes and we call g a weakly interpolating function if weak sign changes may occur, all this for real x .

The L_p -measure as defined for $p \geq 1$ will not be a norm for $p < 1$, since the triangle inequality is no longer valid. However, if the exponent $1/p$ is dropped from the expression, the extremal points will not be changed and the triangle inequality will be valid; however the resulting expression is not homogenous (for proofs, see e.g. 2.12 in [13]). For the rest of this section we will nevertheless speak of such modified expressions as norms, knowing their true properties.

We will now combine the results of Motzkin, Walsh and Rice with those

of the traditional L_p -norms in a table containing entries of the unicity and interpolating properties of best approximations.

	X a finite point set		X an interval	
	uniceness	interpolation	uniceness	interpolation
$1 < p \leq \infty$	unique	strong	unique	strong
$p=1$	non-unique	weak	unique	strong or vanishing on a set of measure > 0
$0 < p < 1$	non-unique	exact	non-unique	weak

The table requires some comments since the characters of the entries are somewhat different. Thus for the two upper rows useful characterizations exist both in terms of interpolating properties and in other ways. Generally speaking, the situation is worse for L_p -approximations with $0 < p < 1$. We will in general not have uniqueness, and characterizations which easily lend themselves to algorithms are absent. However, if X is a finite point set, matters are a little more manageable. From theorem 6 of [6] it follows that if X has $m > n$ points and is to be approximated by an element of an n -dimensional Haar subspace, then best approximations must interpolate to f in at least n points. Since there is only a finite number of ways, $\binom{m}{n}$, to select the interpolating points, we can find a best approximating function by exhausting all possibilities. When we are approximating by polynomials, even more is known since also sufficient conditions for best approximations have been derived in [7].

When X is an interval and $0 < p < 1$, very little is known. A complicated result in [10] concerning polynomial approximation gives a lower bound on the number of zeroes, indicating that zeroes of multiplicities > 1 will be involved. Thus we have in this case

weakly interpolating best approximations, the exact nature of which is not known in contrast to the case of the finite point set. Further results have been obtained in the special case of approximation by a constant on an interval [11]. That the analysis of this seemingly simple case involves formidable difficulties may give some hint of the complexity of the situation.

We will now define L_p -approximation for $p \leq 0$ in any linear subspace of the functions defined on X . It is believed that if the subspace satisfies the Haar condition, the entries in a table extended to this region will be the same as those for $0 < p < 1$. Whether the other results in [6]-[12] can be extended is not known.

6.2. Approximation in the L_p -sense on a real finite point set.

We will here, following a suggestion made by Professor E.W.Cheney, start by defining an order relation between nonnegative m -tuples $u=(u_1, u_2, \dots, u_m)$, $v=(v_1, v_2, \dots, v_m)$. Define $N(u)$ = number of zero components in u . Then

- a) $p=+\infty$ $u \prec_p v$ means $\max_i u_i \leq \max_i v_i$
- b) $1 \leq p < \infty$ $u \prec_p v$ means $\sum_i u_i^p \leq \sum_i v_i^p$
- c) $0 < p < 1$ $u \prec_p v$ means $\sum_{u_i \neq 0} u_i^p \leq \sum_{v_i \neq 0} v_i^p$
- d) $p=0$ $u \prec_0 v$ means $\{N(u) > N(v)\}$ or $\{N(u)=N(v) \text{ and } \prod_{u_i \neq 0} u_i \leq \prod_{v_i \neq 0} v_i\}$
- e) $-\infty < p < 0$ $u \prec_p v$ means $\{N(u) > N(v)\}$ or $\{N(u)=N(v) \text{ and } \sum_{u_i \neq 0} u_i^p \geq \sum_{v_i \neq 0} v_i^p\}$
- f) $p=-\infty$ $u \prec_{-\infty} v$ means $\{N(u) > N(v)\}$ or $\{N(u)=N(v) \text{ and } \min_{u_i \neq 0} u_i \leq \min_{v_i \neq 0} v_i\}$

Now, let X be a prescribed set of n points and f a function defined on it. Let H be any n -dimensional linear subspace of functions defined on X and let $g, h \in H$. Then, for a given p , we say " g is at least as good an L_p -approximation to f as h " if

$$\|f-g\|_p \leq \|f-h\|_p. \quad \text{If this is valid for all } h \in H, \text{ then we call}$$

g a best approximation.

It is believed that the order relations could be further refined to yield unique best approximations in all cases. This is however postponed to another occasion.

The definitions given by the order relations a) - c) will give the usual results while d) - f) are somewhat different in their character since here those points which are used for the construction of the approximation have been excluded from the measure of deviation.

Uniqueness depends on f and it is quite clear that we do not always get unique best approximations in c) - f) as can be seen from the following example. If $X=\{1,2\}$, $f=\{0,1\}$ and we want to approximate by a constant, then 0 and 1 will be as good L_p -approximations for all $p \leq 1$. Some reduction in the number of best approximations could be accomplished by the use of limiting procedures along the lines used in sect. 4.

6.3. Approximation in the L_p -sense on a closed interval.

Let $X=(c,d)$ be a closed interval on the real axis and f a function defined on it. Let H be an n -dimensional subspace of functions defined on X . Assuming the existence of the integrals, we define

$$L_p(f, g) = \int_c^d |f(x) - g(x)|^p dx \quad p \neq 0, p \neq -\infty$$

$$L_0(f, g) = \int_c^d \log |f(x) - g(x)| dx \quad p = 0$$

$$L_{-\infty}(f, g) = \min_{x \in X} |f(x) - g(x)| \quad p = -\infty$$

Here L_0 has been derived in [14] by computing $\lim_{p \downarrow 0} L_p(f)$ as $p \downarrow 0$ which will result in the expression

$M_0(f) = \exp(\int_R \log |f| dV / \text{meas } R)$, but since we are interested only in the minimization of the expression, we retain only the integral. Also in [14], $M_{-\infty}(f)$ is defined as the essential minimum (the infimum disregarding sets of measure zero) which can be used for the definition of $L_{-\infty}(f, g)$.

In our efforts to use these L_p -expressions for the purpose of defining best approximations, we try to get an idea of their interpolation properties by looking at the extrema.

Let $g(a, x) = \sum_{i=1}^n a_i h_i(x)$. For $p=0$, we may look at the second derivative of L_0 with respect to the coefficients a_k :

$$\frac{\partial L_0}{\partial a_k} = \int_c^d \frac{-h_k(x)}{(f(x) - g(a, x))} dx$$

$$\frac{\partial^2 L_0}{\partial a_k^2} = \int_c^d \frac{h_k^2(x)}{(f(x) - g(a, x))^2} dx$$

If $a = (a_1, a_2, \dots, a_n)$ is a set of parameters for which $g(a, x)$ has no root on (c, d) , then the second derivative calculated above exists and is negative. Since the second derivative must be nonnegative for a minimum, we conclude that for an optimal parameter, $g(a, x)$ must interpolate to f in at least one point.

For $p < 0$, we want a maximum of L_p . Taking the derivatives with respect to the coefficients we get

$$\frac{\partial L_p}{\partial a_k} = \int_c^d p |f(x) - g(a, x)|^{p-1} (-h_k(x)) \cdot \operatorname{sgn}(f(x) - g(a, x)) dx$$

$$\frac{\partial^2 L_p}{\partial a_k^2} = \int_c^d p(p-1) |f(x) - g(a, x)|^{p-2} (h_k(x))^2 dx$$

The second derivative is clearly positive in the intervals when the denominator in the integrand has no root. In the same way we get here an interpolating function as a best approximation.

It must be admitted that by these easy procedures we have not established anything comparable to the known results for other cases since the number of zeroes has not been established. However it is believed that results similar to [10] will be obtained when the subspace H satisfies the Haar condition and we will e.g. speak of best polynomial approximations of degree n as having at least $n+1$ (not necessarily different) points in common with the function which is to be approximated.

In order to get more useful definitions we must here, as in the case of the finite point set, exclude the singularities of $1/|f(x) - g(a, x)|$. How this should be done is not clear to the author.

One method of doing it can be illustrated on the $L_{-\infty}$ -approximation case. Let H be any n -dimensional linear subspace of functions defined on X . Then if $g^* \in H$ and

$$\min_{x \in X} |f(x) - g^*(x)| \leq \min_{x \in X} |f(x) - g(x)| \quad \text{for all } g \in H, \text{ then } g^*$$

could be called a best approximation.

If this definition is used, then evidently all $g \in H$ which have any point in common with f on the interval will be best approximations. In order to choose between such approximations g_1 and g_2 , one might ask whether the following inequality is true:

$$\min_{\substack{f(x)=g_1(x) \\ x+\epsilon \in X}} |f(x+\epsilon)-g_1(x+\epsilon)| \leq \min_{\substack{f(x)=g_2(x) \\ x+\epsilon \in X}} |f(x+\epsilon)-g_2(x+\epsilon)|$$

If the inequality is valid for all sufficiently small $\epsilon > 0$, then g_1 is called a best $L_{-\infty}$ -approximation. This will obviously not guarantee uniqueness but we have by such a definition severely restricted the class of best approximations.

Defined in this way, a best $L_{-\infty}$ -approximation in terms of polynomials is clearly a partial sum of the Maclaurin -expansion at a point in the interval if f is analytic there.

A definition (which has to be worked out) in the way hinted at here may have the nice property of relating expansions in terms of a Chebyshev set of functions to the L_p -approximations in terms of these functions. Further, the L_p -approximations, $p < 1$, which have a tendency to have multiple zeroes may be of interest in the context of spline approximations.

We will end this section by calculating some L_p -approximations.

6.4. Examples.

Ex. 1. L_0 -approximation of x^2 on (c,d) by a constant.

Let the constant be a^2 , which is no restriction since it must be nonnegative to have the interpolating property.

$$\begin{aligned} z &= \int_c^d \log|x^2 - a^2| dx = \left[x \log|x^2 - a^2| - 2x + a \log|(x+a)/(x-a)| \right]_c^d = \\ &= d \log|d^2 - a^2| - 2d + a \log|(d+a)/(d-a)| - (c \log|c^2 - a^2| - 2c + a \log|(c+a)/(c-a)|). \end{aligned}$$

$$\begin{aligned} \frac{dz}{da} &= d(-2a)/(d^2 - a^2) + \log|(d+a)/(d-a)| + a/(d+a) + a/(d-a) - \\ &\quad - (c(-2a)/(c^2 - a^2) + \log|(c+a)/(c-a)| + a/(c+a) + a/(c-a)) = \\ &= \log|d+a| - \log|d-a| - \log|c+a| + \log|c-a| \end{aligned}$$

$$\frac{d^2 z}{da^2} = 1/(d+a) + 1/(d-a) - 1/(c+a) - 1/(c-a)$$

$$\frac{dz}{da} = 0 \text{ when } \log \left| \frac{(d+a)(c-a)}{(d-a)(c+a)} \right| = 0$$

i.e. when $(d+a)(c-a) = (d-a)(c+a)$

$$+ \text{ sign: } a(d-c) = 0 \quad a=0 \quad d \neq c$$

$$- \text{ sign: } a^2 = dc \quad a = \pm \sqrt{dc}, \text{ } d \text{ and } c \text{ of the same sign}$$

$$\left(\frac{d^2 z}{da^2} \right)_{a=0} = \frac{2(c-d)}{dc} \quad \begin{array}{llll} \text{which is} & \text{pos} & \text{if} & c < 0 < d \Rightarrow \text{min} \\ & " & " & c < d < 0 \Rightarrow \text{max} \\ & " & " & 0 < c < d \Rightarrow \text{max} \end{array}$$

$$\left(\frac{d^2 z}{da^2} \right)_{a=\pm\sqrt{dc}} = \frac{4}{d-c} \quad \text{which is pos} \quad \Rightarrow \text{min}$$

Thus, if $0 \in (c,d)$ then 0 is the best approximation

" $0 \notin (c,d)$ " cd " " " "

If $(c,d) = (0,1)$ we will have the following approximations to x^2

$$\begin{array}{ll} L_\infty & 1/2 \\ L_2 & 1/3 \\ L_1 & 1/4 \\ L_0 & 0 \end{array}$$

Ex. 2. L_0 -approximation to x^2 on (c, d) by a linear expression $ax+b$. Assume that a best approximation interpolates to x^2 in the points $\alpha_1, \alpha_2 \in (c, d)$.

$$z = \int_c^d \log|x^2 - ax - b| dx = \int_c^d \log|x - \alpha_1| dx + \int_c^d \log|x - \alpha_2| dx =$$

$$= c - d + (\alpha_1 - c) \log(\alpha_1 - c) + (d - \alpha_1) \log(d - \alpha_1) +$$

$$+ c - d + (\alpha_2 - c) \log(\alpha_2 - c) + (d - \alpha_2) \log(d - \alpha_2)$$

$$\frac{\partial z}{\partial \alpha_1} = \log(\alpha_1 - c) + 1 - 1 - \log(d - \alpha_1) = \log((\alpha_1 - c)/(d - \alpha_1))$$

$$\text{analogously } \frac{\partial z}{\partial \alpha_2} = \log((\alpha_2 - c)/(d - \alpha_2)) .$$

$$\frac{\partial^2 z}{\partial \alpha_1^2} = 1/(\alpha_1 - c) + 1/(d - \alpha_1) > 0$$

$$\frac{\partial^2 z}{\partial \alpha_2^2} = 1/(\alpha_2 - c) + 1/(d - \alpha_2) > 0$$

=> minimum

$$\frac{\partial z}{\partial \alpha_1} = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = (d+c)/2$$

$$\frac{\partial z}{\partial \alpha_2} = 0$$

Minimum for $a = (d-c)$
 $b = -(d-c)^2/4$

If $(c, d) = (0, 1)$ we will have the following best approximations to x^2 .

$$\begin{array}{ll} L_\infty & x - 1/8 \\ L_2 & x - 1/6 \\ L_1 & x - 3/16 \\ L_0 & x - 1/4 \end{array}$$

Ex. 3. Polynomial approximation with degree n of $\sin(x)$ on $(0, \pi)$ in the L_∞ -sense.

a) $n=0$

$$z = \min_{x \in X} |\sin(x) - a|$$

The infimum of z is 0, reached when $\sin(x)=a$ (1)
 z is to be small in a neighbourhood of the minimum which
 implies that the first derivative with respect to x should
 be small.

$$\frac{dz}{dx} = \min_x |\cos(x)-0| \quad \text{The infimum of this is 0,}$$

reached when $\cos(x)=0$ (2)

(1)&(2) gives $x=\pi/2$, $a=1$

Best approximation by a constant is 1.

b) $n=1$.

By the same reasoning

$$z = \min_x |\sin(x) - ax - b| \quad \Rightarrow \quad \sin(x) = ax + b \quad (3)$$

First derivative

$$\frac{dz}{dx} = \min_x |\cos(x) - a| \quad \Rightarrow \quad \cos(x) = a \quad (4)$$

Second derivative

$$\frac{d^2z}{dx^2} = \min_x |-\sin(x)| \quad \Rightarrow \quad \sin(x) = 0 \quad (5)$$

$$(5) \text{ gives } \begin{matrix} x_1=0 \\ x_2=\pi \end{matrix} \quad \Rightarrow \text{ by (3), (4) } \begin{matrix} a_1=1 & b_1=0 \\ a_2=-1 & b_2=\pi \end{matrix}$$

Best linear approximations are x and $\pi-x$

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A generalization of L_p approximation to $p \leq 0$.
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Contents:

	p.
1. Introduction	1
2. L_p approximation on a finite point set	4
3. L_p approximation on the interval $[0,1]$	12
4. On algorithms	25
Appendix: The existence of best L_p approximations	38
References	43

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1. Introduction.

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1. The purpose of the present paper is the study of some properties of best p :th power approximations to certain functions defined on a finite point set (" l_p -approximation") or on an interval (" L_p -approximation"). The subject is a classical one for $p \geq 1$, but also for $0 < p < 1$ there is an extensive literature. The central papers of the field are those of Jackson [14-17], Hoel [13], Motzkin and Walsh [21-25, 32-33] and Rice [26, 29]. We will not deal with the case $p \geq 1$; instead we will introduce a generalization which will allow p to take on values on the whole real axis. This type of generalization was suggested by Ekblom and Henriksson in [7], but then in a statistical context. Sketches of the role this generalization could play in approximation theory were given by Henriksson in [11] and Ekblom in [6]. The main object of this paper is the introduction of a generalization of L_p approximation in the direction of a general p and no attempts have been made to minimize the assumptions needed for the results. The outline of the paper is the following. In chapt. 1 definitions common to the discrete and continuous cases are given, while chapt. 2 and 3 deal with those cases separately; however the existence of best approximations is treated in a unified way in the Appendix. Finally, in chapt. 4 algorithmic problems are discussed.

2. Notation and problem definition

We will use a mixture of the various notations introduced by earlier workers in this field.

- Let X be a subset of the real axis consisting either of a finite set of m distinct points $\{x_i\}_{i=1}^m$ or of the closed interval $[0, 1]$.
- Let f be a real-valued function defined on X . For $X=[0, 1]$ we will

first only require that f is continuous, but then raise our requirements on its differentiability. f is the function which is to be approximated.

- Let $F(A, x)$ be the approximating function, a real-valued function defined on X . Here A denotes the parameter vector in some parameter space P , which is normally identical to some R^n .

- Let l_p and L_p be real-valued functions which are used to give generalized measures of distance between the functions f and $F(A)$.

The parameter p is the one that is central to this study and the range of its values will be specified in each case. The values of l_p and L_p will depend on various functions and parameters and these will be written out only when their roles are to be stressed. Let momentarily $\mathcal{L}_p = l_p$ or L_p according to whether X is a finite set or not. We will later discuss for which values of p this function is defined, let us assume that the function is indeed well-defined. The basis of definition will be the general weighted mean of order p .

With this preliminary notation we can formulate the $\mathcal{L}_p$ approximation problem.

Statement of problem

Given X , f and $F(A)$ defined on X and p , determine $A^* \in P$, where P is the parameter space such that

$$\mathcal{L}_p(X, f - F(A^*)) \leq \mathcal{L}_p(X, f - F(A)) \quad \text{for all } A \in P.$$

If such a value A^* can be found we call $F(A^*)$ a best $\mathcal{L}_p$ approximation to f on X .

We call $F(A^*)$ a locally best approximation to f if there is a neighbourhood $N(A^*)$ of A^* in P such that for all $A \in N(A^*)$ the above inequality is satisfied.

3. Further notation

Let H_n be a n -dimensional linear subspace of functions $h_i, i=1, 2, \dots, n$ continuous on X . We call H_n a Haar space or equivalently, $\{h_i\}_{i=1}^n$ a Chebyshev set if 0 is the only function in H_n which has n or more roots on X . An equivalent condition is that for every selection of n distinct points $x_j \in X$, the determinant $\det(h_i(x_j)) \neq 0$.

A more restricted concept is that of ET-systems (Karlin and Studden in [18]) or extended Chebyshev systems which fulfill the condition that for every selection of k distinct points $x_r \in X, r=1, 2, \dots, k$ $k \leq n, \alpha_r \geq 0, \sum_{r=1}^k (\alpha_r + 1) = n$, we have that

$$\begin{vmatrix} h_1(x_1) & h_2(x_1) & \dots & h_n(x_1) \\ h_1^{(1)}(x_1) & h_2^{(1)}(x_1) & \dots & h_n^{(1)}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(\alpha_1)}(x_1) & h_2^{(\alpha_1)}(x_1) & \dots & h_n^{(\alpha_1)}(x_1) \\ h_1(x_2) & h_2(x_2) & \dots & h_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(\alpha_2)}(x_2) & h_2^{(\alpha_2)}(x_2) & \dots & h_n^{(\alpha_2)}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(\alpha_k)}(x_k) & h_2^{(\alpha_k)}(x_k) & \dots & h_n^{(\alpha_k)}(x_k) \end{vmatrix} = \det(h_i^{(v)}(x_j)) \neq 0$$

A further condition on H_n is that the basis functions should be analytic on X . If also this condition is fulfilled, we denote the function space by H_n^* . We will use $L(A)$ to denote an element from such a linear Haar space of the appropriate type. The conditions above are clearly fulfilled by polynomials $h_i(x) = x^{i-1}, i=1, 2, \dots, n$, but other such systems exist.

2. l_p approximation on a finite point set.

1. "The general weighted means of order p ", M_p , are here defined as follows (Hardy-Littlewood-Polya [10])

Let $\{b_i\}_{i=1}^m$, $b_i > 0$. Let $w_i > 0$, $\sum_{i=1}^m w_i = 1$.

Then define

$$M_p(w, b) = \left(\sum_{i=1}^m w_i b_i^p \right)^{1/p} \text{ for } |p| < \infty, p \neq 0.$$

If we introduce the obvious generalizations

$$\begin{aligned} M_p(w, b) &= \prod_{i=1}^m b_i^{w_i} && \text{for } p = 0 \\ &= \min_i b_i && \text{for } p = -\infty \\ &= \max_i b_i && \text{for } p = +\infty \end{aligned}$$

which can be derived with the use of limiting procedures, $M_p(w, b)$ is well-defined for $-\infty \leq p \leq +\infty$.

If we allow $b_i = 0$ for some i 's, no trouble is caused for $p > 0$.

However, when $-\infty < p < 0$, we can see from

$$M_p(w, b) = \left(\sum w_i b_i^p \right)^{1/p} = \frac{1}{\left(\sum w_i \left(\frac{1}{b_i} \right)^{-p} \right)^{-1/p}}$$

that we will have terms of the type $(1/0)^{-p}$ in the denominator.

The classical interpretation of this is to say that $M_p(w, b) = 0$ if $p < 0$ and $b_i = 0$, some i . We note that this holds for $p = 0$ and $p = -\infty$ as well, implying that with this modification $M_p(w, b)$ is a well-defined quantity for $b_i \geq 0$, $-\infty \leq p \leq +\infty$.

2. When using $M_p(w, b)$ for purposes of approximation, we can treat the b_i as parameters, while p and w_i are fixed. We note that the problem of minimizing M_p will have the same solution in terms of b as the following

$$\begin{aligned} \min_b \sum_i w_i b_i^p & \quad 0 < p < +\infty \\ \max_b \sum_i w_i (1/b_i)^{-p} & \quad -\infty < p < 0 \end{aligned}$$

Simply expressed: we may drop the exponent $1/p$ if we take note of its sign. Similarly, for $p=0$ we may just as well seek

$$\min_b \sum_i w_i \log b_i$$

These alternative expressions will be used from time to time.

We will also define some interpolation properties on the set $X = \{x_i\}_{i=1}^m$.

A function g ($\neq 0$) defined on X is said to have n a) strong b) weak c) exact interpolations on X if there are points $x_j \in X$, $x_j < x_{j+1}$ such that

$$\begin{aligned} \text{a)} \quad & g(x_j) \cdot g(x_{j+1}) < 0 & j = 1, 2, \dots, n \\ \text{b)} \quad & g(x_j) \cdot g(x_{j+1}) \leq 0 & \text{"-"} \\ \text{c)} \quad & g(x_j) = 0 & \text{"-"} \end{aligned}$$

For $g = f - F(A)$ fulfilling such conditions to a specified number n , we say that $F(A)$ is a a) strongly b) weakly c) exactly interpolating function to f on X .

The definitions have been taken over from Rice [29] and are here just reworded.

3. We may now define our approximation problem. We will normally not be interested in the value of the generalized mean $M_p(w, b)$ itself, but the value of b which makes it a minimum; to stress this and to conform to the usual notation we will use the term l_p approximation.

Definition of best l_p approximations.

Let $X = \{x_i\}_{i=1}^m$ be a finite point set and f a real-valued function defined on it, let $F(A)$ be a member of some function space defined on X , with A belonging to a parameter space P . Let $w = (w_i), i=1, 2, \dots, m, w_i > 0, \sum_i w_i = 1$, let $M_p(w, b)$ be a function defined as above for $-\infty \leq p \leq +\infty$ then determine an $A^* \in P$ by the minimization of

$$l_p(f - F(A)) = M_p(w, |f - F(A)|), \text{ where }$$

$f - F(A)$ denotes the residual vector with components $f(x_i) - F(A, x_i)$

$i = 1, 2, \dots, m$. If we can find a value $A^* \in P$ such that

$$l_p(f - F(A^*)) \leq l_p(f - F(A)) \quad \text{for all } A \in P$$

then we call $F(A^*)$ a best l_p approximation to f on X .

4. The case $1 \leq p \leq +\infty$ is well-known, the case $0 < p < 1$ somewhat less so. We will turn to the case $-\infty \leq p \leq 0$ which we believe has not been discussed earlier.

In general, the approximating function $F(A)$ and the parameter space P may be such that we will not necessarily have any A_0 with

$$f(x_i) - F(A_0, x_i) = 0 \text{ for some } i \text{ and thus get } l_p(f - F(A)) = 0.$$

But in most realistic problems we can and in view of the non-negativity of the l_p -function it is then obvious that we get a best approximation for such an A_0 . The existence of best approximations is then trivial as remarked in the Appendix and as stated in the following theorem

Theorem 1. If $F(A)$ is such that for some $A_0 \in P$ and some i $f(x_i) - F(A_0, x_i) = 0$, then $F(A_0, x)$ is a best l_p approximation for any $p \leq 0$ and any set of weights.

Proof: With the assumptions given, then by definition we have

$$l_p(f - F(A_0)) = 0$$

since this case corresponds to one residual = 0. On the other hand

$$l_p(f - F(A)) \geq 0, \quad \text{all } A.$$

Hence, $l_p(f - F(A_0)) \leq l_p(f - F(A))$, all $A \in P$, which is what defines a best l_p approximation. //

Corollary. Let $p \leq 0$. If $F(A) \equiv L(A)$, i.e. if the approximation is from a linear Haar space, best l_p approximations will interpolate exactly at least once. Conversely any $L(A)$ which interpolates exactly to f on X at least once is a best l_p approximation.

The equivalence stated in the corollary is a very weak one compared with what is known about best approximations from Chebyshev sets when $p > 0$ (in which cases we have statements on interpolation in n points, not just one). From this we may learn that the fact that some function is a best l_p approximation with $p \leq 0$ is a very weak property. On the other hand, when we have strong assumptions on F , such as $F(A, x) \equiv L(A, x)$, we can use the freedom to impose stronger conditions. This is done in the next section.

5. We first need some further notation.

Let $X_A^0 = \{ \text{points } x \in X \mid f(x) - F(A, x) = 0 \}$

and let X_A^1 be the complement of X_A^0 in X .

Let $W(A) = \sum_{x_i \in X_A^0} w_i$, i.e. the sum is taken over the points of

exact interpolation.

Definition Let $p \leq 0$ and let $F(A_0, x)$ and $F(A, x)$ be best l_p approximations.

If $[W(A_0) > W(A)]$

or

$$[W(A_0) = W(A) \text{ and } l_p(X_A^1, f(x) - F(A_0, x)) \leq l_p(X_A^1, f(x) - F(A, x))]$$

for all $A \in P$, then we call $F(A_0, x)$ a strictly best l_p approximation.

The definition is based on the naive and intuitive approach that the summed weights of the singular terms in the denominator of the original expression $l_p(X, f(x) - F(A, x))$ should count first since a higher sum of these weights may be said to produce a steeper zero of the l_p function than a lower one. The definition of strictly best approximations must obviously be arbitrary to a large extent; the question is for what purpose the freedom should be used. The intention here

is simply to get an analogue to the theory developed for $p > 0$, as will be seen in the following theorems.

Theorem 2. Let $p \leq 0$ and $L(A, x) \in H_n$. Then, strictly best approximations will interpolate exactly to f in n points.

Proof: Assume that $L(A', x)$ is a strictly best l_p approximation to f which interpolates to f in $k < n$ points, say on the set $X' \in X$. From the properties of Chebyshev sets we can construct a function $L(A'', x)$ which interpolates to f on $X' \cup X''$, where X'' is any set of $n-k$ points in X but not in X' .

But $W(A'') = W_{X' \cup X''}(A'') = W_{X'}(A'') + W_{X''}(A'') = W_{X'}(A') + W_{X''}(A'') > W_{X'}(A') = W(A')$ since $W_{X''}(A'')$ is greater than 0 (X'' is not empty and the weights are positive).

The inequality $W(A'') > W(A')$ is however in contradiction to the assumption that A' was a strictly best approximation. Hence $k \geq n$. //

Theorem 3. Given $p \leq 0$, there exists a set of positive weights $\{w_i\}_{i=1}^m$ for each function $L(A, x) \in H_n$ which interpolates to f in n points on X , which will make this function a strictly best l_p approximation.

Proof: Let X' be the set of n points which are defined by the assumptions. Define ϵ by $1/(n+1) > \epsilon > 0$ and put

$$w_i = 1/(n+1) + \epsilon/n \quad \text{for } x_i \in X'$$

$$w_i = 1/[(n+1)(m-n)] - \epsilon/(m-n) \quad \text{for } x_i \notin X'$$

Clearly $\sum_i w_i = n(1/(n+1) + \epsilon/n) + (m-n)[1/((n+1)(m-n)) - \epsilon/(m-n)] = 1$.

Then $W(A') \geq n/(n+1) + \epsilon$.

Let Y' ($X' \subseteq Y'$) be the set of all the points on which exact interpolation in fact takes place.

Now let $L(A'', x)$ be any other function $\in H_n$ and Y'' the set of points of exact interpolation to f . If $Y' \subseteq Y''$, then since functions in H_n are fully determined by n points, $Y' = Y''$ and $L(A', x) \equiv L(A'', x)$.

If $X' \not\subseteq Y''$, then there is at least one point in X' which is not in Y'' . Since the points in X' have weights with values $1/(n+1) + \epsilon/n$, then

$$W(A'') \leq 1 - (1/(n+1) + \epsilon/n) = n/(n+1) - \epsilon/n$$

Therefore $W(A') > W(A'')$ and no strictly better approximation has been found. Since A'' was arbitrary we have shown that with the given weights $L(A', x)$ is a strictly best approximation. //

As seen from theorems 2 and 3 we get an equivalence between strictly best l_p approximations ($p \leq 0$) and functions which interpolate exactly to f in n points. This is what was aimed at in the definition of strictly best approximations.

The restriction of the number of feasible solutions of the l_p approximation problem which was imposed by our definition naturally raises the question whether this process could be continued to achieve uniqueness for some "strictly strictly" best approximation. Such an endeavour, however, seems futile in view of the following example.

example: $m = 2$, $w_1 = w_2 = 0.5$, $(x, f(x)) = \{(1, 1), (2, 2)\}$,

approximation by a constant. No obvious way of selecting between the two strictly best approximations $c=1$ and $c=2$ seems to exist.

A definition, which dealt only with the case of equal weights, but otherwise was similar to the one presented here, was used by the author in [11].

One final remark may be added to the definition part. When for some $p \leq 0$ and two candidates for strictly best approximation A'' and A' , we have that $W(A') = W(A'')$, then the sums of the residuals raised to the power of p determines which one of A' and A'' is best. However, one could just as well use the power q with $0 < q \leq +\infty$. Such

interpolating approximation might be called $l_{p,q}$ approximation (or l_q^- approximation, since only the non-positivity of p is used) and have a strange mixture of properties. However we will not here go further in this direction.

6. For the sake of completeness we will now include the results for $p \leq 0$ in the larger setting of l_p approximations.

Theorem 4. Let $\{h_i\}_{i=1}^n$ be a Chebyshev set and X a finite point set. Let $L(A, x) = \sum_i a_i h_i(x)$. Then, given $-\infty \leq r \leq 0 < s < 1 < t \leq +\infty$ we have the following pairs of identical sets:

- (a) $\{A \mid L(A, x) \text{ is a best weighted } l_t \text{ approximation to } f(x)\} =$
 $\{A \mid L(A, x) \text{ interpolates strongly to } f(x) \text{ } n \text{ times on } X\}$
- (b) $\{A \mid L(A, x) \text{ is a best weighted } l_1 \text{ approximation to } f(x)\} =$
 $\{A \mid L(A, x) \text{ interpolates weakly to } f(x) \text{ } n \text{ times on } X\}$
- (c) $\{A \mid L(A, x) \text{ is a best weighted } l_s \text{ approximation to } f(x)\} =$
 $\{A \mid L(A, x) \text{ interpolates exactly to } f(x) \text{ } n \text{ times on } X\}$
- (d) $\{A \mid L(A, x) \text{ is a best weighted } l_r \text{ approximation to } f(x)\} =$
 $\{A \mid L(A, x) \text{ interpolates exactly to } f(x) \text{ once on } X\}$
- (e) $\{A \mid L(A, x) \text{ is a strictly best weighted } l_r \text{ approximation to } f(x)\} =$
 $\{A \mid L(A, x) \text{ interpolates exactly to } f(x) \text{ } n \text{ times on } X\}$

Proof: (a), (b), (c) are directly taken over from Rice [29]. (d) is the Corollary of this chapter and (e) consists of theorems 2 and 3. //

Corollary. Let $L(A, x)$, X and $f(x)$ be as given in theorem 4. Then for any $r, s, t, u : -\infty \leq r < s \leq 0 < t' < u < 1$, the following sets are identical:

- (1) $\{A \mid L(A, x) \text{ is a strictly best weighted } l_r \text{ approximation to } f(x)\},$
- (2) $\{A \mid \text{ " " " " " " } l_s \text{ " " " " }\}$
- (3) $\{A \mid \text{ " " " best " } l_t \text{ " " " " }\}$
- (4) $\{A \mid \text{ " " " " " " } l_u \text{ " " " " }\}$

Proof: The equivalence between (1) and (2) comes from (e) in theorem 4, between (3) and (4) was derived by Rice in [29], between (2) and (3) from (c) and (e) in theorem 4. //

3. L_p approximation on the interval $[0,1]$.

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1. For the interval case the means M_p take the following forms, as discussed by Hardy-Littlewood-Polya in [10] and Beckenbach - Bellman in [4].

Let $X=[0,1]$, let g be continuous on X with $g(x) \geq 0$. Let w be continuous on X with $w(x) > 0$ and $\int_X w(x)dx = 1$. Then define

$$M_p(w,g) = \left[\int_X w(x) g(x)^p dx \right]^{1/p} \quad \text{for } |p| < \infty, p \neq 0.$$

$$M_p(w,g) = \exp\left(\int_X w(x) \log(g(x)) dx\right) \quad \text{for } p = 0$$

$$M_p(w,g) = \min_{x \in X} g(x) \quad \text{for } p = -\infty$$

$$M_p(w,g) = \max_{x \in X} g(x) \quad \text{for } p = +\infty$$

The cases $p = \pm\infty, p=0$ are again derived with the use of limiting procedures.

For $p \leq 0$ and with $g(x)=0$ for some $x \in X$, the integral in the definition may not converge. We will adopt the usual convention and say that $M_p(w,g) = 0$ in such cases.

We will define the following interpolation properties. Let X denote a closed interval. A function $g \neq 0$ defined on X is said to have n

a) strong b) weak c) exact d) Hermitian interpolations on X

if there are points $x_j \in X, x_j < x_{j+1}$ such that

$$a) \quad g(x_j) \cdot g(x_{j+1}) < 0 \quad j=1,2,\dots,n$$

$$b) \quad g(x_j) \cdot g(x_{j+1}) \leq 0 \quad \text{"-"}-$$

$$c) \quad g(x_j) = 0 \quad \text{"-"}-$$

$$d) \quad g^{(s)}(x_j) = 0 \quad j=1,2,\dots,k; \quad s=0,1,\dots,\alpha_j; \quad \sum_{j=1}^k (\alpha_j+1) = n.$$

For $g = f - F(A)$ fulfilling such conditions to a specified number n

we say that $F(A)$ is a a) strongly b) weakly c) exactly

d) Hermitian interpolating function to f on X .

It is seen that strong interpolation implies weak and that exact implies Hermitian (named so because of its correspondence to full Hermitian interpolation [12]). For a continuous g we also have that strong interpolation implies exact interpolation.

Further, we note that the minimization of $M_p(w, g)$ is equivalent to

$$\min_{g \in G} \int_X w(x) g(x)^p dx \quad 0 < p < \infty$$

$$\min_{g \in G} \int_X w(x) \log(g(x)) dx \quad p = 0$$

$$\max_{g \in G} \int_X w(x) g(x)^p dx \quad -\infty < p < 0$$

where G is the relevant function space.

We also note that $M_p(w, g)$ is a non-negative, continuous, non-decreasing function of p [4].

2. We can now define best L_p approximations for the interval case.

Definition. Let f and $F(A)$ be continuous on $X=[0,1]$, let w be a weight function on X , let A belong to a parameter space P , let $-\infty \leq p \leq +\infty$, then define

$$L_p(f - F(A)) = M_p(w, |f - F(A)|)$$

If we can find an $A^* \in P$ such that

$$L_p(f - F(A^*)) \leq L_p(f - F(A)) \quad \text{for all } A \in P,$$

then we call $F(A^*)$ a best L_p approximation to f on X .

3. The definition is analogous to the common one for $p > 0$. The case $1 \leq p \leq \infty$ is well-known and will not be discussed here. The case $0 < p < 1$ has been treated by Jackson in [16,17], by Hoel in [13], by Motzkin and Walsh in [23-25,32,33] and by Rice in [26,29]. Here, we prefer to treat this case together with the case $p \leq 0$, an interval which we believe has not been discussed earlier. As will be seen $p=0$ and $p=1$ do not constitute the kind of boundary points between the various cases of interpolation as they do for a finite set. The

boundary points will here rather be given by properties of the residual function $f - F(A)$. However, to establish existence we have to impose very restrictive conditions; in fact we will assume that $F(A) = L(A)$ and that f and $L(A)$ are analytic on X .

For the problem of existence, we will sum up the pertaining results of the Appendix in the following theorem.

Theorem 1. Let $f \in A[X]$, let $L(A) \in H_n^*$. Then for $-\infty \leq p \leq +\infty$ there exists best L_p approximations.

Proof: The result is a classical one for $0 < p \leq +\infty$.

The analyticity of f and the basis functions of H_n^* guarantees that $f - L(A)$ never has a zero on X of infinite order without vanishing identically. Hence from theorem 2.3 in the appendix we have existence of best approximations for $-\infty < p \leq 0$. For $p = -\infty$ existence is trivially clear. //

4. Necessary conditions for best approximations.

We will now turn to a result corresponding to the well-known orthogonality property of best L_p approximations, $p \geq 1$. The result is a weak one - no conditions on sufficiency have been established. We will make a small extension to the result of Hoel [13].

Theorem 2. Let $f \in A[X]$, let $L(A) \in H_n^*$. Assume that the zeroes of $f - L(A)$ are at most of order k for any A . Then if $p \leq p_0 = -1/k$, $L(A^*)$ is a best L_p approximation iff

$$I = \int_X |f(x) - L(A^*, x)|^p w(x) dx$$

is divergent.

Proof: The result is immediate since $\inf_A M_p(w, |f - L(A)|) = 0$ is reached for $A = A^*$ and this corresponds to the divergence of the integral. //

Theorem 3. Let $f \in A[X]$, let $L(A) \in H_n^*[X]$, let $p < 1$. Assume that the

zeroes of $f - L(A)$ are at most of order k . Then a necessary condition for $L(A^*, x)$ to be a best L_p approximation to f on X is that

$$J = \int_X w(x) \frac{|f(x) - L(A^*, x)|^p}{|f(x) - L(A^*, x)|} L(A, x) dx$$

either

- a) for all A is convergent and $= 0$
- b) at least for some A is divergent

Case b) may occur only when $p \leq p_1 = 1 - 1/k$.

Proof: Hoel [13] has shown that if the integral

$$J' = \int_X \frac{|f(x) - L(A^*, x)|^p}{|f(x) - L(A^*, x)|} w(x) dx$$

is convergent, then $J=0$.

If J' is not convergent, then the cause of this must be a zero of $f - L(A^*)$ of order $r \leq k$ at, say x_0 .

Then the integral

$$\int_Y w(x) (f(x) - L(A^*, x))^{p-1} L(A, x) dx$$

where Y is an interval with one boundary point at x_0 and such that $f - L(A^*) > 0$ on Y , must be divergent too, e.g. for an A such that $L(A, x) \neq 0$ on Y . But this is true only when $(p-1)r \leq -1$, i.e. when $p \leq 1 - 1/r \leq 1 - 1/k$. //

We may comment that since $L(A) \in H_n^*$, we know that $k \geq n$.

5. Interpolating properties of best approximations.

The two preceding theorems will now be used to determine the character of interpolation of best approximations.

Theorem 4. Let $p \leq p_0 = -1/k$, k being defined as before. Then if $L(A^*)$ is a best L_p approximation to $f \in A[X]$ on X , it interpolates exactly to f in a point with a multiplicity $r \geq -1/p$.

Proof: From theorem 2 we know that for $p \leq p_0$ the integral

$$I = \int_X w(x) |f(x) - L(A^*, x)|^p dx \quad \text{is divergent.}$$

If $f-L(A^*)$ has a zero of order at most r , the divergence takes place only if $p \cdot r \leq -1$, i.e. if $r \geq -1/p$. //

Theorem 5. Let $1 > p > p_1 = 1-1/k$. If $L(A^*) \in H_n^*$ is a best L_p approximation to $f \in A[X]$ on X , then $L(A^*)$ has n strong interpolations to f .

Proof: From theorem 3 we know that for a best approximation the integral

$$J = \int_X w(x) \frac{|f(x) - L(A^*, x)|^p}{|f(x) - L(A, x)|^p} L(A, x) dx = 0 \text{ for all } A.$$

To reach a contradiction, assume that $L(A^*)$ interpolates strongly to f $r < n$ times. Then we can find a value of A such that $L(A, x)$ is zero in the r points of exact interpolation implied by the strong interpolation. Further it can be made such that the integrand in J is greater than 0 on X . But this contradicts the assumption that $J=0$ and hence that $L(A^*)$ is a best approximation. Hence $r \geq n$. //

Theorem 6. Let $p_1 \geq p > p_0$, where $p_1 = 1 - 1/k$, $p_0 = -1/k$, the zeroes of $f - L(A)$ being at most of order k . Then if $L(A^*) \in H_n^*$ is a best L_p approximation to $f \in A[X]$ on X , then $L(A^*)$ has n Hermitian interpolations to f on X .

Proof: For $0 < p < 1$, Jackson [17] has shown that the added multiplicities of the zeroes of $f-L(A^*)$ will reach at least n . In fact Jackson's argument can be applied also when $p_0 < p \leq 0$, i.e. as long as $I = \int_X w(x) |f(x) - L(A^*, x)|^p dx$ is convergent.

For if the summed multiplicities were $r < n$, then we could construct a function $L(B) \in H_n^*$ which would give the same zeroes with the same multiplicities and the same sign of $f-L(B)$ as $f-L(A^*)$ where this function is different from zero. Then $(f-L(A^*))/L(B)$ would be positive on X and have a minimum t there. Then $L(A^*) - t \cdot L(B)$ would be uniformly closer to f than $L(A^*)$ on X . Hence we would have found a better L_p approximation in contradiction to the assumption. Therefore $r \geq n$ and the theorem is proved. //

Theorems 4, 5 and 6 together give a picture of the interpolation properties of best L_p approximations for $p < 1$. As seen, the roles played by p_0 and p_1 parallel those of 0 and 1 in the finite case; these are the values of p for which a change of type of interpolation may take place.

However, the value of p for which changes actually take place will depend on the situation at hand. In [24], a paper entirely devoted to the approximation problem

$$\min_c \int_0^1 w(x) |x^\alpha - c|^p dx, \quad \alpha > 0, \quad 0 < p < 1, \quad w(x) = 1,$$

Motzkin and Walsh have shown by a most thorough analysis that $c=0$ is the best L_p approximation if $\alpha \geq 2/(1-p)$, hence for $p \leq 1 - 2/\alpha$, while the general bound derived above would give that this could happen when $p \leq 1 - 1/\alpha$.

For $p > 0$, stronger theorems than those above have been proved, theorems giving more detailed information on the multiplicities of the roots. The reader is referred to Hoel [13] and Walsh and Motzkin [32, 33] for such theorems.

For theorems of the converse type, i.e. conditions under which interpolating functions are best L_p approximations, the situation is quite different. This is not surprising in view of the absence of a characterization theorem giving also sufficient conditions. A preliminary study indicates that converse theorems can be established if we no longer demand that w is continuous on X but only integrable. This would however be a break with the classic properties of weight functions and we do not give any results in this direction.

We will finally collect the results in the following theorem.

Theorem 7. Let $X=[0,1]$, let $L(A)$ be a linear combination from an extended Chebyshev set with analytic basis functions on X , having dimension n . Let f be analytic on X and let k be the maximum order of any zero of $f - L(A)$ for any A . Let weight functions be positive and continuous on X . Then, given r, s, t, u

$$-\infty \leq r \leq p_0 < s \leq p_1 < t < 1 \leq u \leq +\infty$$

where $p_0 = -1/k$, $p_1 = 1 - 1/k$,

the following conditions prevail

- (a) The best weighted L_u approximation to f has n strong interpolations and is unique.

Any strongly interpolating function $L(A)$ is a unique best weighted L_u approximation to f .

- (b) Best weighted L_t approximations to f have n strong interpolations to f .

- (c) Best weighted L_s approximations to f have n Hermitian interpolations to f .

- (d) Best weighted L_r approximations to f have an interpolation point on X of order $\geq \max(1, [-1/r])$.

Any function $L(A)$ with an interpolation point to f of multiplicity $\geq \max(1, [-1/r])$ is a best weighted L_r approximation to f .

Proof: (a) is a classic result

(b) follows from theorem 5

(c) follows from theorem 6

(d) follows from theorem 4, if $[-1/r]$ is interpreted as the smallest integer $\geq -1/r$, the order of interpolation being integral since the functions are analytic. For $p = -\infty$, we need the 1. //

6. On strictly best approximations.

It is natural to ask whether it is possible to define something similar to the strictly best l_p approximations given for $p \leq 0$

and finite point sets. The same properties of extended Chebyshev systems can obviously be used to impose restrictions among best approximations. One possible approach will briefly be discussed here.

Let $-\infty \leq p \leq p_0$.

Define $W(A) = \sum_{x_A} \alpha_k w(x)$, where $x_A = \{x | f(x) - L(A, x) = 0\}$

i.e. the sum is taken over the points for which exact interpolation takes place and the terms of the sum are the weights multiplied by the multiplicities of the roots. $W(A)$ as defined here is in direct correspondence to the definition in the discrete case. Let

(a) $[W(A^*) > W(A)]$

(b) for $p > -\infty$ $[W(A^*) = W(A)$ and $\int_X u(A^*, x) w(x) |f(x) - L(A^*, x)|^p dx$
 $\geq \int_X u(A, x) w(x) |f(x) - L(A, x)|^p dx]$

where $u(A, x)$ is a function with zeroes in the points of interpolation of order large enough for the integrals to converge

$$\text{for } p = -\infty [W(A^*) = W(A) \text{ and } \min_{\xi_1} (f(x) - L(A^*, x))^{(n)}_{x=\xi_1} \leq \min_{\xi_2} (f(x) - L(A, x))^{(n)}_{x=\xi_2}]$$

where ξ_1 denotes roots of multiplicity n .

If $L(A^*, x)$ is a best L_p approximation and a) or b) holds for all A which are best approximations, then we call A^* a strictly best L_p approximation.

As seen, in order to get a definition using the L_p integral, we have introduced additional weight functions with zeroes of appropriate order at the points of interpolation. The arbitrariness in the choice of these functions is evident and certainly leaves doubt about the meaningfulness of the definition. It is, however, enough for our purposes to note that since best approximations are first determined by $W(A)$, they can be defined such that $f - L(A^*)$ has n Hermitian interpolations.

7. Interpolating functions and juxtapolynomials.

We now turn to a subject which "is apparently different from either best approximations in a norm, or interpolation" (Rice) but which nevertheless is closely related to best p th power approximations for $p > 0$. Our aim is to show that these relations hold also for $p \leq 0$.

From Rice [29] we get the following definitions

Definition. Let $X=[0,1]$ and let a function f defined on X be given.

Then $L(A_1, x)$ is closer to $f(x)$ than $L(A_2, x)$ if

$$|L(A_1, x) - f(x)| < |L(A_2, x) - f(x)|$$

for those $x \in X$ such that the right hand side of the inequality is positive and if $L(A_1, x) = L(A_2, x)$ for those $x \in X$ such that $L(A_2, x) = f(x)$.

Definition. $L(A^*, x)$ is a juxtapolynomial to $f(x)$ on X if there exists no polynomial closer to $f(x)$ than $L(A^*, x)$ on X .

For a finite point set the following theorem is valid.

Theorem (Rice). Let f be given on a finite point set. Then the following sets of linear approximating functions are identical

- (a) $\{L(A) \mid L(A) \text{ is a juxtapolynomial to } f \text{ on } X\}$
- (b) $\{L(A) \mid L(A) \text{ weakly interpolates to } f \text{ } n \text{ times on } X\}$

With the results of chapter 2 in mind we might have added

- (c) $\{L(A) \mid L(A) \text{ is a (strictly) best } l_p \text{ approximation to } f \text{ on } X, -\infty \leq p \leq +\infty\}$

The beauty of this theorem naturally raises the question whether something similar holds for the interval case. However no such direct correspondence is known and indeed Rice uses the following example to show non-equivalence:

$f(x) = x^2$, $X=[0,1]$. There is no constant which is uniformly closer to f than 0, hence 0 is a juxtapolynomial to f . But 0 is not a best L_1 approximation for any positive continuous

weight function.

We may comment that 0 is a best L_0 approximation for $w(x)=1$ and the question is whether new insight into the problem can be reached with the introduction of L_p approximations $p \leq 0$. We have the following theorem [29].

Theorem (Rice). A necessary and sufficient condition that the polynomial $P_n(x)$ be a juxtapolynomial to a continuous function f on $[0,1]$ is that $f - P_n$ has zeroes on $[0,1]$ with a total sum of multiplicities of at least n .

It is obvious that for $p > p_0$ best L_p approximations are juxtapolynomials as is also the case for strictly best L_p approximations, $p \leq p_0$ if the tentative definition in the preceding section is accepted. No full equivalence between juxtapolynomials and best L_p approximations, $-\infty \leq p \leq +\infty$, can however be reached in the absence of the converse theorems for $p_0 < p < 1$.

8. Continuity of coefficients.

One property of best approximations may be of special value for the subject of algorithms. We have the following result from Hoel [13].

Theorem (Hoel). Let $L(A) \in H_n$. If

$$\int w(x) |f(x) - L(A, x)|^p dx \quad \text{is minimized by } A = A_p,$$

then the coefficients vector A_p is a continuous function of p , for every value of $p > 0$.

Since in general we have several best approximations, Hoel uses a special interpretation of continuity: for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every q for which $|q-p| < \delta$ every coefficient in $L(A_q, x)$ will differ from those of someone of the $L(A_p, x)$ by less than ϵ .

The theorem can in fact be extended to all $p > p_0$. We start by generalizing a lemma by Hoel.

Lemma. Let $L(A) \in H_n^*[X]$, let $f \in A[X]$. The absolute values of the coefficients A_p , being a best L_p approximation, have an upper bound independent of p for all $\alpha \leq p \leq \beta$, where α and β are any two numbers such that $p_0 < \alpha < \beta$.

Proof: By lemma 2.1 in the appendix we have that

$$L_p(f - L(A)) = \left[\int_X w(x) |f(x) - L(A, x)|^p dx \right]^{1/p}$$

is a continuous function of A and that $L_p(f - L(B))$ is continuous and $\neq 0$ in a neighbourhood of $\epsilon = 0$ if $p > -1/n$. But here we have $p > -1/k \geq -1/n$. We have that

$$L_p(f - L(A_p)) = |A| \cdot L_p(f/|A| - L(B)) \quad , \quad B = A/|A| .$$

Since $L_p(f - L(A_p)) \leq L_p(f - L(A))$, all A , by definition of A_p and since $L_\alpha(f - L(A)) \leq L_p(f - L(A)) \leq L_\beta(f - L(A))$ by the monotonicity of the L_p function, we have

$$|A_p| = \frac{L_p(f - L(A_p))}{L_p(f/|A_p| - L(B_p))} \leq \frac{L_p(f - L(A_p))}{L_\alpha(f/|A_p| - L(B_p))} \leq \frac{c(\beta)}{c(\alpha)}$$

where $c(\beta)$ depends only on β , $c(\alpha)$ depends only on α and is different from zero since $p > p_0$. //

Theorem 8. Let $L(A) \in H_n^*[X]$, let $f \in A[X]$. If $p > p_0$ the coefficients of best approximations are continuous functions of p in the multi-valued sense.

Proof: By lemma 2.1 in the appendix we have that $L_p(f - L(A))$ is a continuous function of A .

Now, using Hoel's argument: Let q_i be a sequence such that $\lim_{i \rightarrow \infty} q_i = p$.

Then for each q_i there are one or several A_{q_i} which may be thought of as points in an n -dimensional space. From the preceding lemma these points are bounded and thus gives rise to limit points as $i \rightarrow \infty$. Since we have continuity, $\lim_{i \rightarrow \infty} L_p(f - L(A_{q_i})) = L_p(f - L(A_p))$. The limit points can be identified with those giving the limit

values. Hence, in the multiple valued sense, $\lim_{q \rightarrow p} A_q = A_p$. //

Theorem 9. The preceding theorem is true also when $p \rightarrow p_0$.

Proof: Let A_p be a best L_p approximation and A_{p_0} a best L_{p_0} approximation. Let q_i be a sequence such that $\lim_{i \rightarrow \infty} q_i = p_0$, $q_i > p_0$.

Since A_{q_i} represents a full Hermitian interpolation of order n ,

A_{q_i} satisfies the n th order linear equations system

$$\{h_j^{(v)}(x_k)\} A_{q_i} = f^{(v)}(x_k), \text{ where the matrix of coefficients}$$

is the one of an extended Chebyshev system.

But since h_i and f are analytic on X , $h_i^{(v)}$ and $f^{(v)}$ are bounded quantities, and hence A_{q_i} is bounded. Therefore, as $q_i \rightarrow p_0$, there are one or several limit points.

If all limit points of $A_{q_i} = \text{some } A_{p_0}$, there is nothing to prove since we then have continuity in the multiple-valued sense.

If some limit point of A_{q_i} , say A' , is different from any A_{p_0} , then $f - L(A')$ can have a zero of at most order $r < -1/p_0$, hence

$p_0 r > -1$ and $L_{p_0}(f - L(A'))$ is continuous and $\neq 0$.

But we have $L_p(f - L(A_p)) < L_p(f - L(A_{p_0}))$.

When $p \rightarrow p_0$ the right member $\rightarrow 0$ while the left does not.

A contradiction has been reached and hence we have continuity to the right at p_0 . //

9. Two modest examples.

To simply illustrate the proposed type of approximations, we will give two modest examples.

example 1 Find the best L_p approximations to x^2 on $[0,1]$ by a constant. We have $n=1$, $k=2$ since x^2-a has a zero of order 2 when $a=0$.

$w(x)=1$ $p_0=-1/2$, $p_1=1/2$.

Some best approximations are easily calculated by minimization of

the integral $\int_0^1 |x^2-a|^p dx$

p	a_p
$+\infty$	0.50
2	0.33
1	0.25
0.5	0.16
$0 \leq p < -0.5$	0
$-0.5 \leq p \leq -\infty$	$0 \leq a_p \leq 1$

example 2 Approximation of $x^{1/2}$ on $[0,1]$ with a constant. The assumptions of the theorems regarding analyticity are not fulfilled here.

The L_p -integral converges for $p > -1$:

$$\int_0^1 |x^{1/2} - a|^p dx = 2[a(a^{p+1} + (1-a)^{p+1})/(p+1) - (a^{p+2} - (1-a)^{p+2})/(p+2)], p \neq 0$$

Some best approximations

p	a_p
∞	0.50
2	0.67
1	0.71
0.1	0.77
0	0.78
-0.1	0.79
-0.5	0.85
-0.95	0.98

4. On algorithms.

1. The problem of finding algorithms for the computation of best l_p and L_p approximations has lately been the subject of a large number of papers. However, with one exception, only the area $p \geq 1$ has been covered. We will discuss the algorithmic problems shortly and not give a complete listing of suggested methods, but rather a selection which shows the kind of techniques that are used for $p \geq 1$. Our aim by this discussion is to investigate whether any of these techniques are transferable to $p < 1$ and further to examine some new approaches to the algorithmic problems when $0 < p < 1$.

2. To facilitate the discussion we will first introduce a classification scheme for the methods. To be more specific, we assume $p > 0$ and hence that the problem is that of minimization.

We will use the term direct methods for those which attack the problem directly, without use of any reformulation or use of the equivalences between the actual problem and some other approximation problem. The methods here differ mainly in the way characterization theorems are used. By indirect methods we mean such ones that use any of the available ways of reformulation into an approximation problem (with some other value of p) which is easier to solve. The assumed proximity of best $L_{p+\Delta p}$ approximations to those of L_p is one of the possible approaches here. Another one is that of using theorems of equivalence between L_p and L_q approximations ($p \neq q$), which are accomplished by the use of different weight functions. We will call such methods weight function methods.

3. Direct methods.

For some values of p , finite direct methods are available. For $p=2$ we have the well-known normal equations which are easily solved, at least by use of adequate numerical methods. For $p=\infty$ and $p=1$, the discrete linear approximation problem has been shown to be equivalent to

linear programming problems and good algorithms have been published: for $p = \infty$, Goldstein and Cheney [9] and Barrodale and Young [2], for $p = 1$, Rice [27] and Barrodale and Young [2]. A L_1 descent method has been proposed by Usow in [30,31]. For $1 < p < \infty$ the approximation problem has been formulated as one of convex programming by Barrodale and Roberts in [3]. For the case L_∞ , the existence of a strong characterization theorem allows special methods, e.g. those of Remes where attention is concentrated to the residuals [5].

Standard optimisation methods may be used, at least for $1 < p < \infty$ but the implications of the following remark by Fletcher-Grant-Hebden in [8] are obvious: "There has been little investigation into satisfactory general techniques for the calculation of best L_p approximations for a function and often use is made of a standard optimisation process such as Davidon's method. For the particular case of least squares approximation, the use of such a standard process is obviously inefficient as the optimal parameters are better determined by setting up and solving the normal equations." It is easy to agree that the special properties which are present in the approximation problem at hand should be used for the construction of an algorithm.

4. Indirect methods.

The classic method making use of the continuity of the solutions in terms of p is Polyas algorithm (see e.g. Cheney [5]). Here, solutions to a series of L_{2p} approximation problems are calculated and as p goes to infinity the solution converges to the L_∞ solution. The normal equations (or rather the equations given by the orthogonality conditions) can here be solved with Newton's method. A similar approach is used by Karlowitz for $2 < p \leq \infty$ in [14].

For $p = 1$, Abdelmalek [1] calculates the l_{1+2} - k solutions, for $k=0,1,\dots$ which converge to the l_1 solution. For the normal equations the method of Fletcher-Powell is used.

Weight function methods were introduced by Lawson [20] for $p = \infty$.

A series of weight functions are generated which converge to one which makes a l_∞ problem equivalent to a l_2 problem weighted with this function. The method was generalized by Rice and Usow to that of any p , $2 < p \leq \infty$ [28].

Another weight function approach is that of Fletcher-Grant-Hebden which essentially rewrites

$$\int w(x) |f(x) - L(A, x)|^p dx \quad \text{as} \quad \int w(x) |f(x) - L(A, x)|^{p-2} [f(x) - L(A, x)]^2 dx$$

This is then treated as a weighted L_2 problem which is solved iteratively. The iteration by itself is divergent but with a modification convergence takes place for $2 < p < \infty$, as shown in [8]. The method is in fact an adaptation of the Newton-Raphson method. Essentially the same method was recently suggested by Kahng [35].

5. Algorithms for $p < 1$.

For the interval case, we do not know of any algorithmic work in this area. The absence of discussion is not very surprising since in practical work the integrals nearly always have to be replaced by sums in one form or another. We will just make a few tentative remarks on this case.

For $1 > p > p_1$, the orthogonality equations are necessary conditions for best approximations and they can be solved with traditional methods, as for $p \geq 1$. However, it seems reasonable to expect numerical difficulties; computations of integrals with a pointwise singular integrand are not trivial and the integrand is strongly oscillating. On the other hand we may be able to get adequate starting points using the continuity of coefficients for p , i.e. we can start with a solution

to a more easily solved problem and then go stepwise downwards.

For $p_0 < p \leq p_1$ it seems more reasonable to work with some weight function method, using a series expansion in terms of the approximating function as a starting solution.

For the discrete case two methods, one direct and one indirect, have been proposed, both by Barrodale - Roberts in [3]. The l_p approximation problem is formulated as a problem in mathematical programming; we get linear constraints and an object function which has local minima for each set of interpolation points and even if these are used to form a piecewise linear function, it may still have several local minima. Barrodale-Roberts remark: "Thus although the optimal solutions are located at extreme points of the convex set of constraints, there appears to be no efficient way of descending to an optimal point from an extreme point of which a local minimum occurs."

The two main characteristics of the discrete problem are thus the following:

Global minima are found at some point sets belonging to a finite set, whence they always can be reached by enumeration technique.

The function to be minimized has first derivatives which are discontinuous at all points belonging to the set of possible solutions.

The second of these characteristics must under the circumstances be thought of as an inherent difficulty of the problem; its presence makes most of the methods of general optimization inapplicable. Nor can the methods developed for $p > 1$ be used, at least not without strong modifications.

The attention here will be concentrated to investigations of the case of discrete approximation for $0 < p < 1$. There are two reasons for this. First there is only one report in the literature on work in this area and hence results on new work may be of interest. Secondly we believe that an understanding of this case must be reached before further advances can be made. In consequence with this we judge it desirable to report also on approaches and methods that have failed

The two methods of Barrodale-Roberts may be characterized as follows: For the direct method, a modification of the well-known Barrodale-Young l_1 algorithm [2] is used, such that a local minimum of the l_p function is reached. Of course there is no guarantee that this is also a global minimum. For the indirect method essentially the same process is used, only the calculation is performed for $p_i = 1 - \Delta p \cdot i$, $i=0,1,\dots$ until the desired value of p is reached. If just one point of interpolation is exchanged between successive p_i 's this can be successfully handled by the algorithm. For a small Δp few dramatic changes between best approximations are expected to occur and we get a behavior which is similar to the continuity of solutions in the interval case. However, optimal solutions are not at all guaranteed, as elegantly shown in the following figure, taken from [3].

The example in the figure shows the situation when a best l_p approximation in form of a straight line is to be computed on a set of four points. The best approximation will then interpolate exactly in two points, i and j . For the $\binom{4}{2}=6$ possible combinations of points, the measures of distance $S_{ij}(p) = (\sum |f_k - L(A, x_k)|^p)^{1/p}$ are given. Best approximations are then defined by the lower envelope of the curves, $\inf_{i,j} S_{ij}(p)$. Since the algorithm only can handle intersections where just one interpolation point is exchanged (such as those at A and C), it will miss the exchange at B and give a nonglobal minimum for the

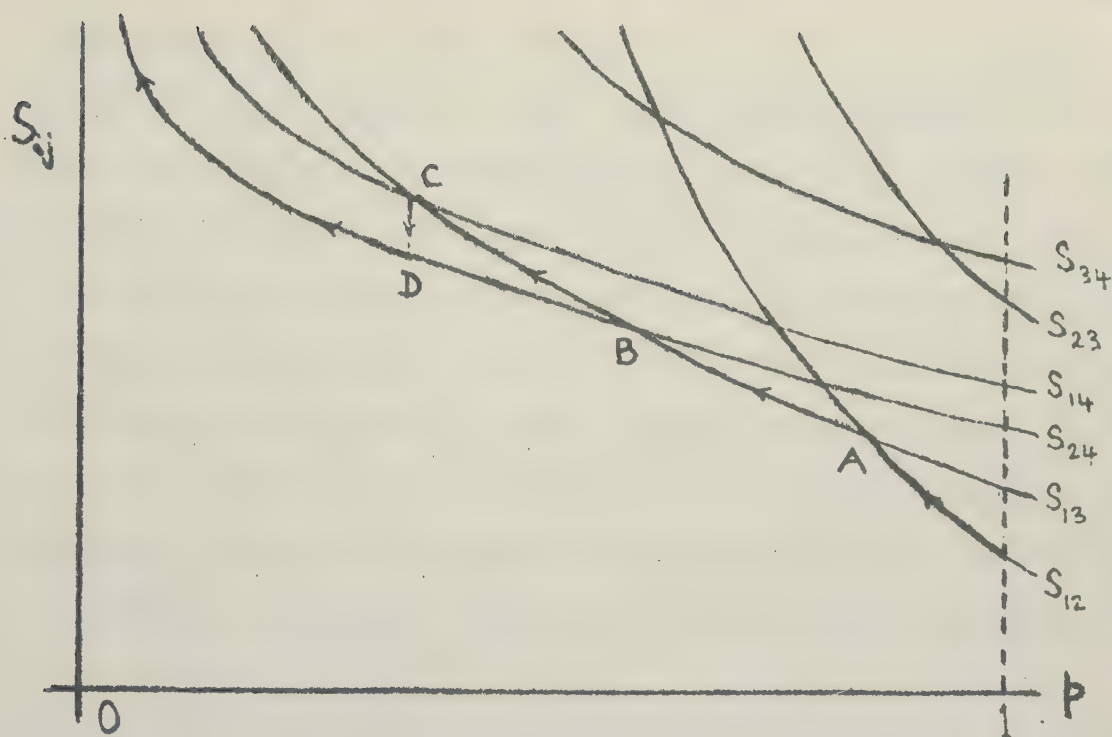


Fig.1. The S_{ij} -curves are shown as functions of p for a problem where $m=4$, $n=2$.

p -interval between B and D.

Reluctance towards the use of a multistep calculation in terms of p , as required by the algorithm of Barrodale-Roberts, has given rise to the following idea. For values of p not too small, let the curves $S_{ij}(p)$ be approximated by their tangents S'_{ij} at $p=1$.

$$\begin{aligned} \frac{dS_{ij}}{dp} &= \left[-\log \left(\sum_k |f_k - L(A, x_k)|^p \right) / p^2 + \frac{\sum_k |f_k - L(A, x_k)|^p \log |f_k - L(A, x_k)|}{p \cdot \sum_k |f_k - L(A, x_k)|^p} \right] \cdot \\ &\quad \cdot \left[\sum_k |f_k - L(A, x_k)|^p \right]^{1/p} = \\ &= -S_{ij}(p) \left[\log S_{ij}(p) - \frac{\sum_k |f_k - L(A, x_k)|^p \log |f_k - L(A, x_k)|}{\sum_k |f_k - L(A, x_k)|^p} \right] / p \\ \text{Hence, } S'_{ij}(p) &= S_{ij}(1) + (1-p) S_{ij}(1) \left[\log S_{ij}(1) - \frac{\sum_k |f_k - L(A, x_k)| \log |f_k - L(A, x_k)|}{\sum_k |f_k - L(A, x_k)|} \right] \end{aligned}$$

Clearly this is a rather cumbersome expression, almost comparable in

computational expense with the one it is supposed to replace.

However it can be used in a two-stage "pruning" procedure as follows:

Given a reasonably good approximation $S^*(p)$ to $\inf_{i,j} S_{ij}(p)$, we may for each candidate check whether $S_{ij}(1) > S^*(p)$; if this is the case then the candidate can securely be dismissed since S_{ij} is a decreasing function of p . If the first screening is passed we can calculate and test $S'_{ij}(p)$ and then, only if it is necessary, $S_{ij}(p)$.

However we have to report a negative result for this approach to the problem: too few combinations could be dismissed from the set of solutions at the first stage and little if anything was gained.

Barrodale-Roberts report that for over 90 % of the test examples their second method converges to the true optimal solution (which, given time, money and patience, always can be found by checking the $\binom{m}{n}$ possible combinations of interpolation points). The indirect algorithm is reported to be more successful than the direct one.

To further demonstrate the complex character of the function to be minimized/maximized, we give the following one-dimensional example.

Ex.1 Find $\min_c \sum_{i \neq j} |f(x_i) - c|^p$, where $f(x_i), i=1, 2, \dots, 12$ are given below. We know that $c=f(x_j)$, some j , hence $\sum_{i \neq j} |f(x_i) - f(x_j)|^p$ are given in the table.

Local optimum values have one underlining, global ones have two.

x_j	1	2	3	4	5	6	7	8	9	10	11	12
$f(x_j)$	1	2	3	4	20	25	30	70	72	74	76	100
p												
1	465	455	447	441	377	<u>367</u>	<u>367</u>	447	455	467	483	723
0.5	62.6	61.3	<u>60.7</u>	60.8	60.2	<u>59.8</u>	61.0	64.3	<u>64.0</u>	64.7	66.4	86.2
0.1	15.0	14.9	<u>14.8</u>	14.9	15.3	<u>15.2</u>	15.4	15.3	<u>15.2</u>	15.2	15.4	16.5
-0.1	8.25	8.37	<u>8.39</u>	8.30	7.99	<u>8.00</u>	7.92	8.03	<u>8.11</u>	8.10	8.00	7.36
-0.5	3.48	3.91	<u>3.94</u>	3.53	2.38	<u>2.43</u>	2.26	2.73	<u>3.02</u>	3.01	2.71	1.52
-2.0	1.37	2.25	<u>2.26</u>	1.37	0.06	<u>0.09</u>	0.05	0.34	0.57	<u>0.57</u>	0.34	0.01

6. An iterative approach.

We will here discuss an approach to the problem, somewhat similar to that of Fletcher-Grant-Hebden in [8] for $2 < p < \infty$. The problem is attacked at two levels, one local and one global.

First we will make the formal rewriting

$$(1) \quad \sum_i w_i |f(x_i) - L(A, x_i)|^p = \sum_i w_i \underbrace{|f(x_i) - L(A, x_i)|^{p-1}}_{u_i(A)} |f(x_i) - L(A, x_i)|$$

The idea would be to use this formula for iterative solution, finding A_{k+1} by solving the l_1 problem

$$(2) \quad \min_A \sum_i w_i' |f(x_i) - L(A, x_i)| \quad \text{where } w_i' = w_i \cdot u_i(A_k).$$

We may first note that, as written, the weights at the interpolation points are infinite and hence makes the iteration stationary. This can be dealt with in various ways - the type of approximation to be made for these terms does not seem to be very sensitive. The method which so far has been found most successful was simply to let the w_i' keep their values from the previous iteration. The whole process is started by setting $u_i = 1$, thus making the l_1 solution the starting point for the iteration.

A method like the proposed one, which is discrete, does not easily lend itself to a convergence analysis. The type of iteration used indicates a linear convergence, at least in an initial state, to a local minimum with a convergence factor proportional to $(1-p)$. This belief is not contradicted by the experimental results.

The second problem is that the above process at best can be expected to reach the vicinity of a local minimum which is not necessarily a global one. After a minimum has been reached one or several restarts of the procedure is needed to find out whether there are other local minima with equal or lower function values. Several methods to avoid

reaching the same point have been tested; so far the one found best was to keep the weights w_i for those points not in the interpolation set while those in it are initially set = 0.

An interesting way of getting away from a local minimum once reached is given by the following iterative approach, different from (1). We make the following formal rewriting

$$(3) \quad \sum_i w_i |f(x_i) - L(A, x_i)|^p = \sum_i w_i |f(x_i) - L(A, x_i)|^{\lambda p} |f(x_i) - L(A, x_i)|^{(1-\lambda)p}$$

$$0 < \lambda < 1$$

Then instead of minimizing the original expression, we minimize over A to find an A_{r+1}

$$(4) \quad \sum_i w_i u_i |f(x_i) - L(A, x_i)|^{(1-\lambda)p} \text{ where } u_i = |f(x_i) - L(A_r, x_i)|^{\lambda p},$$

A_r being the local minimum found in the previous iteration.

By this we have transformed the original problem to a "worse" one, $(1-\lambda)p$ being smaller than p and hence in general making the problem have a more nonlinear behaviour. Use of this procedure on the one-dimensional test example (ex. 1) gave the following observations.

1. The procedure (with $\lambda=0.5$) when used with a perfect algorithm (in fact the "naive" algorithm of a total check and $p=0.9, 0.7, 0.5, 0.3, 0.1$ managed to locate the one to three local minima of the function after which oscillations between them took place. Also for negative values of p (with a negative value of λ) similar oscillations took place.

2. The procedure when used with the imperfect algorithm for finding local minima which we earlier described, did not work. The double iteration implied never started in the test example since the solutions never got away from a good starting solution.

Hence, this method cannot be regarded as useful for its purpose. The situation may be reconsidered if it can be found to work with other methods of local minimization.

Results of a test.

The proposed method can be formally written in the following way.

local iteration

find A_{k+1} from $\min_i \sum w_i u_i(A_k) \cdot |f(x_i) - L(A_{k+1}, x_i)|$

where

$$u_i(A_k) = \begin{cases} |f(x_i) - L(A_k, x_i)|^{p-1} & \text{if } x_i \notin X_{\text{int}, A_k} \\ u_i(A_{k-1}) & \text{if } x_i \in X_{\text{int}, A_k} \end{cases}$$

X_{int, A_k} denoting the interpolation set of the solution at the k th step of iteration.

Terminate the iteration when a repetition of solutions (cycling) takes place and denote the best approximation found by A^* .

global iteration

apply the above iteration procedure and use as starting values

the first time $u_i(A) = 1$

at subsequent times

$$u_i(A) = \begin{cases} u_i(A^*) & \text{if } x_i \notin X_{\text{int}, A^*} \\ 0 & \text{if } x_i \in X_{\text{int}, A^*} \end{cases}$$

Terminate the iteration when a repetition of solutions occurs or after a fixed number of steps.

The method was tested on the test example provided by Barrodale and Roberts in [3].

ex. 2. Approximate

$$f(x) = \begin{cases} e^x & \text{on } [0, 1] \\ e^{-x} + e^{-1} & \text{on } [1, 2] \end{cases}$$

on $x = 0(0.1)2$ by $L(A, x) = \sum_{j=1}^5 a_j x^{j-1}$

Here $m=21$, $n=5$. Best l_p approximations are computed for $p = 0.1(0.1)0.9, 0.95$

The problem is a tricky one as may be expected from the underlying function with discontinuous derivatives at $x=1$. Also this is reflected in the success rate reported in [3]; the ascent method

succeeds in 6 of the 10 cases as compared to the 90 % of successes reported for test examples in general.

The algorithm as proposed here was written in Algol 60 and run on the Univac 1108 of the Lund University computing centre. For the l_1 computation the simplex algorithm [2], modified to include weights, was used. For each call to the l_1 algorithm a set of interpolation points is produced and the corresponding function value is calculated. Convergence to one interpolation set or repetition of solutions in a small cycle (both equally common) was used to stop the local iteration. Two restarts were made in the global iteration.

The coefficients for the best approximations are given in the table below, while the results of the test are given in table 2 on the next page. The global minima were found in 8 of the 10 cases, missing

p	a_1^*	a_2^*	a_3^*	a_4^*	a_5^*
0.1	1.05034	0.27442	2.98295	-2.49516	0.55725
0.2	1.05034	0.27442	2.98295	-2.49516	0.55725
0.3	1.07057	-0.00180	3.78752	-3.16637	0.72728
0.4	1.07057	-0.00180	3.78752	-3.16637	0.72728
0.5	1.07057	-0.00180	3.78752	-3.16637	0.72728
0.6	1.06362	0.05408	3.94450	-3.38535	0.79144
0.7	1.06490	0.03548	4.00935	-3.44711	0.80859
0.8	1.06490	0.03548	4.00935	-3.44711	0.80859
0.9	1.06455	0.04074	3.98929	-3.42348	0.80097
0.95	1.06455	0.04074	3.98929	-3.42348	0.80097
1.0	1.09359	-0.31307	4.66321	-3.82879	0.87473

Table 1. Best l_1 approximations on $x=0(0.1)2$ by $L(A,x) = \sum_{j=1}^5 a_j x^{j-1}$
to $f(x) = \begin{cases} e^{x^2} & \text{on } [0,1] \\ e^{-x} + e - e^{-1} & \text{on } [1,2] \end{cases}$ (after Barrodale-Roberts [3]).

p	min S	i _{int}	reached by ascent alg.	reached by present algorithm	reached at no.init:no.step	number of iter. before minimum was encountered
0.1	11.293	2 7 16 18 21		(11.416 at 2 4 7 17 20)	2:3	12
0.2	8.2058	2 7 16 18 21		(8.2677 at 2 5 8 15 20)	1:6	6
0.3	6.0859	2 5 8 15 20	*	*	2:6	15
0.4	4.5475	2 5 8 15 20		*	1:6	6
0.5	3.4467	2 5 8 15 20		*	1:5	5
0.6	2.6360	2 4 14 18 20	*	*	2:3	11
0.7	2.0125	2 4 9 14 20	*	*	2:2	7
0.8	1.5484	2 4 9 14 20	*	*	3:2	10
0.9	1.2001	2 4 9 18 20	*	*	2:3	8
0.95	1.0590	2 5 9 18 21	*	*	1:3	5
(1.0	0.9324	2 5 9 18 21)				

Table 2. Results of test runs of example 2. * indicates that the true global minimum was reached, i_{int} denotes the indices of the points in the best interpolation set.

those of $p=0.1, 0.2$. Figures indicating the convergence rate are given in the two rightmost columns and as a further remark it can be noted that the initial function value (given by the l_1 solution) was generally in error of 5 % of the true global minimum, while after two local iterations none of the values were in error of more than 2 %. This of course does not necessarily imply that the coefficients of the proposed approximations or the interpolation sets were close to the true minimum: normally two to four of the five points in the best interpolation set were present.

As seen the algorithm involves extensive and far from trivial computations. Thus at each iteration in search for a local minimum a linear programming problem with $n+1$ variables and $m+2$ linear constraints has to be solved using the simplex method. Further, adequate testing of it involves a large amount of work since all results have to be checked against the set of all the $\binom{m}{n}$ possible solutions.

For comparison with the algorithm of [3] it must be noted that the one given here is more time-consuming, probably by the order of a factor 5 to 10. Further judgment of its merits must wait until more experience is gained. Also more work on it is needed to assure reliability of stopping criteria etc., even if a true global minimum does not have to be guaranteed.

Appendix.

The existence of best L_p approximations.

We shall here treat approximations on finite sets and intervals in a more unified way than in the main text. The following arguments were suggested by L. Hörmander.

Let X be a compact metric space and dm a positive measure on X with total mass 1. If f is a real-valued measurable function on X one defines the mean $M_p(f)$ for $p \neq 0$ by

$$(1) \quad M_p(f)^p = \int_X |f|^p dm$$

It is well known that this is an increasing function of p . If $M_p(f)^p$ is finite for some $p \neq 0$, then the limit $M_0(f)$ of $M_p(f)$ as $p \rightarrow 0$ exists and is given by

$$(2) \quad \log M_0(f) = \int_X \log |f| dm.$$

From Fatou's lemma it follows that

$$(3) \quad M_p(f) \leq \liminf_{j \rightarrow \infty} M_p(f_j) \quad \text{if } f_j \rightarrow f \text{ a.e. and } p > 0,$$

$$(4) \quad M_p(f) \geq \overline{\lim}_{j \rightarrow \infty} M_p(f_j) \quad \text{if } f_j \rightarrow f \text{ a.e. and } p < 0.$$

When the sequence f_j has a uniform upper bound the inequality (4) remains valid for $p = 0$. Also recall that

$$(5) \quad M_p(f+g) \leq M_p(f) + M_p(g), \quad p \geq 1$$

$$(6) \quad M_p(f+g)^p \leq M_p(f)^p + M_p(g)^p, \quad 0 < p < 1.$$

It is well known that for arbitrary measurable $f, f_1, f_2, \dots, f_n$ the function $M_p(f - \sum_{j=1}^n t_j f_j)$ of $t \in \mathbb{R}^n$ has a minimum value if $p > 0$. For comparison with the case $p < 0$ we recall a proof. There is nothing to prove if $M_p(f - \sum_{j=1}^n t_j f_j)$ is $+\infty$ for all t so excluding this case and replacing f by $f - \sum_{j=1}^n s_j f_j$ for suitable s we may assume that $M_p(f) < \infty$. It is no restriction either to assume that $\sum_{j=1}^n t_j f_j$ never vanishes a.e.

if $t \neq 0$ for otherwise we can drop some function f_j . Then there is a constant $c > 0$ such that

$$M_p\left(\sum_{j=1}^n t_j f_j\right) \geq c \quad \text{if } |t| = \sum_{j=1}^n |t_j| = 1.$$

In fact, $M_p\left(\sum_{j=1}^n t_j f_j\right)$ has a minimum value when $|t| = 1$ in view of the semi-continuity implied by (3) and by hypothesis it cannot be 0. Now we conclude using (5) and (6) that

$$\begin{aligned} M_p\left(f - \sum_{j=1}^n t_j f_j\right) &\geq |t|c - M_p(f) \quad , \quad p \geq 1, \\ M_p\left(f - \sum_{j=1}^n t_j f_j\right)^p &\geq |t|^p c^p - M_p(f)^p \quad , \quad 0 < p < 1, \end{aligned}$$

which shows that $M_p\left(f - \sum_{j=1}^n t_j f_j\right) \rightarrow \infty$ as $t \rightarrow \infty$. The semi-continuity (3) then implies that a minimum value is attained.

When $p \leq 0$ the preceding argument breaks down at several places: the lower semi-continuity (3) is replaced by the upper semi-continuity (4) and there is no substitute for (5), (6) from which we can conclude that the function $M_p\left(f - \sum_{j=1}^n t_j f_j\right)$ is large at infinity. We shall therefore have to introduce very restrictive conditions.

At first we only assume that $f, f_1, \dots, f_n$ are bounded. If there is some point $x \in X$ with $m(\{x\}) > 0$ and $f_j(x) \neq 0$ for some j , the approximation problem is trivial for $p \leq 0$. In fact, $M_p\left(f - f(x)/f_j(x) \cdot f_j\right) = 0$ then, if $p \leq 0$. On the other hand, if all f_j vanish at every point where m has a positive mass the problem is not substantially changed if we drop the atomic part of m and normalize the remaining measure. It is therefore no restriction to consider only diffuse measures.

We shall specialize still further and assume that $X = [0, 1] \subset \mathbb{R}$ and that $dm = w(x) dx$ where dx is the Lebesgue measure and w a positive continuous function on X . Sufficient conditions for continuity of $M_p(f)$ are then given by

Lemma 1. Let $f \in C^j(X)$ and assume that $f, \dots, f^{(j)}$ have no common zero in X .

If $p \leq 0$ and $pj > -1$, it follows that

$$C^j(X) \ni g \Rightarrow M_p(g)$$

is continuous in a neighbourhood of f .

Proof: By Lemma 3.3 in Svensson [34] we have for g in a neighbourhood of f

$$m(\{x; |g(x)| < \epsilon^j\}) < C \cdot \epsilon,$$

where C is independent of g and ϵ . If $p < 0$ it follows that

$$\int_{|g(x)| < \delta} |g(x)|^p dm(x) \leq C \int_0^\delta t^p d(t^{1/j}) = C_1 \delta^{p+1/j} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Since it is obvious that

$$\int (|f(x)|^p - |g(x)|^p) dm(x) \rightarrow 0 \text{ as } g \rightarrow f, \text{ it follows that}$$

$$\lim_{g \rightarrow f} |M_p(f)^p - M_p(g)^p| \leq 2C_1 \delta^{p+1/j}$$

Since δ is arbitrary the assertion follows. //

Theorem 2. Let $p < 0$. Then the function $C^k(X) \ni f \Rightarrow M_p(f)$ is continuous if and only if $pk \leq -1$.

Proof: Assume first that $pk \leq -1$. If $f \in C^k(X)$ and $M_p(f) = 0$ it follows from (4) that we have continuity at f . On the other hand, if $M_p(f) \neq 0$ and j is the largest integer with $jp > -1$, we have $j \leq k$ and $f, f', \dots, f^{(j)}$ cannot vanish at any point x_0 . In fact, this would imply that $f(x) = O(|x-x_0|^{j+1})$ and since $p(j+1) \leq -1$ we would conclude that $M_p(f) = 0$ against the hypothesis. The continuity at f now follows from Lemma 1.

Assuming now that $pk > -1$ we shall show that $M_p(f)$ is not even continuous on straight lines in C^k . To do so we choose a number α with $\alpha p > -1$ and $\alpha > k$. Let f be a modification of the function x^α defined as follows for $x > 0$: In the interval $1/2 < x < 2$ we shall have $f > 0$; we set $f(x) = x^\alpha$ near the end points and $f(x) = 1$ for x in a neighbourhood of 1. Furthermore f is C^∞ for $x > 0$ and $f(4x) = 4^\alpha f(x)$, $x > 0$. This implies that $f^{(j)}(4x) = 4^{\alpha-j} f^{(j)}(x)$ so if $j < \alpha$ it follows that $f^{(j)} \rightarrow 0$ at 0. Hence $f \in C^k(X)$. Furthermore $M_p(f) > 0$ since $\alpha p > -1$. However, $M_p(f-t) = 0$ if $t = 4^{-\alpha j}$, $j = 1, 2, \dots$ which proves that $M_p(f-t)$ is not a continuous function of t at $t = 0$. //

Theorem 3. Let $f, f_1, f_2, \dots, f_n \in C^\infty$ and assume that the zeroes of $\sum_{j=1}^n t_j f_j$ are at most of order $k-1$ when $t \neq 0$. Then $\inf_t M_p(f - \sum_{j=1}^n t_j f_j)$ is attained if $-1 < pk < 0$ or if $p=0$ and $f - \sum_{j=1}^n t_j f_j$ never has a zero of infinite order without vanishing identically. If on the other hand $pn \leq -1$ we can choose t so that $M_p(f - \sum_{j=1}^n t_j f_j) = 0$.

Proof: Let first $-1 < pk < 0$. If for some t , $M_p(f - \sum_{j=1}^n t_j f_j) = 0$, then there is nothing to prove. If $M_p(f - \sum_{j=1}^n t_j f_j) \neq 0$, all t , then $f - \sum_{j=1}^n t_j f_j$ can at most have a zero of order $r-1$, where $r < -1/p$; hence we know from lemma 1 that $M_p(f - \sum_{j=1}^n t_j f_j)$ is continuous in t .

Further, we have

$$M_p(f - \sum_{j=1}^n t_j f_j) = |t| \cdot M_p(f/|t| - \sum_{j=1}^n s_j f_j), \quad s = t/|t|.$$

Again from lemma 1 we have that for $|s|=1$ and small ϵ , the function $M_p(\epsilon f - \sum_{j=1}^n s_j f_j)$ is continuous and $\neq 0$. Hence $M_p(f - \sum_{j=1}^n t_j f_j) \rightarrow \infty$ as $t \rightarrow \infty$, which proves that the infimum is attained. When $p=0$ the argument remains valid with obvious modification since $f - \sum_{j=1}^n t_j f_j$ is assumed to have no zero of infinite order.

Then let $pn \leq -1$. We first note that the assumptions imply that the functions f_j are linearly independent on X . This implies that the vectors $(f_j(x), \dots, f_j^{(n-1)}(x))$, $j=1, 2, \dots, n$ are linearly independent at some point in X for otherwise f_j , $j=1, 2, \dots, n$ would in some interval satisfy a differential equation of order $n-1$ in contradiction to the assumptions. Hence we can choose t such that $f - \sum_{j=1}^n t_j f_j$ has a zero of order n on X which proves the last statement. //

Remark: If f_j , $j=1, 2, \dots, n$ form an extended Chebyshev system, then $k=n$ and the two statements in the theorem guarantee the existence of best approximations.

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